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APPLICATIONS OF CONE THEORY TO BOUNDARY VALUE PROBLEMS

by

Gerald Diaz

A DISSERTATION

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Under the Supervision of Professor Allan C. Peterson

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APPLICATIONS OF CONE THEORY

TO BOUNDARY VALUE PROBLEMS

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University of Nebraska, 1989

Adviser: Allan C. Peterson

We are concerned with the existence and comparison of eigenvalues for the eigenvalue problem $(-1)^{n-1}Lu = \lambda P(t)u$, $Tu = 0$, where $Tu = 0$ are appropriate boundary conditions at points in the interval $[a, b]$. Here $u(t)$ is an m-column vector function, $P(t)$ is a continuous $m \times m$ matrix function on $[a, b]$ and $Lu = u^{(n)} + p_1(t)u^{(n-1)} + \cdots + p_n(t)$. ~~We will~~ assume that the corresponding scalar equation $Ly = 0$ is right disfocal on $[a, b]$. ~~We get our~~ existence and comparison results by using several abstract theorems from cone theory in a Banach space.

We first consider the boundary value problem $u^{(n)} + r(t)u = 0$, $u^{(i)}(a) = 0$, $i = 0, 1, \ldots, k-1$ and $u^{(i_j)}(b) = 0$, $j = 1, 2, \ldots, n-k$. Using comparison theorems for Green's functions due to Peterson and Ridenhour we are able to apply cone theory to get the exsitence and uniqueness of an eigenvector in a cone. Further, we can give comparison results between the smallest positive eigenvalues of different eigenvalue problems.

We also examine the n-point right focal eigenvalue problem $(-1)^{n-1}Lu = \lambda P(t)u$, $u^{(i-1)}(t_i) = 0$, for $i = 1, 2, \ldots, n$. Assuming that $Ly = 0$ is right disfocal we give an explicit form for the Green's function. Under certain sign conditions

on the Green's function and conditions on $P(t)$, we can show the existence of a smallest positive eigenvalue. And with further conditions on $P(t)$, that its corresponding eigenvector is essentially unique with respect to a 'cone'. We also have comparison results for the eigenvalue problem above and the problem $Lu = \Lambda Q(t)u$, $u^{(i-1)}(t_i) = 0$ for $i = 1, 2, \ldots, n$. We close this chapter by giving examples where the Green's function has the desired sign conditions. We also give results for the difference equation analog on this problem.

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Introduction

We are concerned with the existence and comparison of eigenvalues for the eigenvalue problem $(-1)^{n-1}Lu = \lambda P(t)u$, $Tu = 0$, where $Tu = 0$ are appropriate boundary conditions at points in the interval $[a, b]$. Here $u(t)$ is an m-column vector function, $P(t)$ is a continuous $m \times m$ matrix function on $[a, b]$ and $Lu = u^{(n)} + p_1(t)u^{(n-1)} + \cdots + p_n(t)$. We will assume that the corresponding scalar equation $Ly = 0$ is right disfocal on $[a, b]$, that is, there does not exist a nontrivial solution y of $Ly = 0$ and points $a \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq b$ so that $y^{(i-1)}(t_i) = 0$ for $i = 1, 2, \ldots, n$.

To get our existence and comparison results, we use several important theorems from cone theory. Krasnosel'skiǐ [12] discovered that if an operator M maps a cone, $\mathcal{P}$, back into itself, and there exists a nontrivial u in $\mathcal{P}$ and an $\varepsilon > 0$ so that $Mu \geq \varepsilon u$, where '$\geq$' means that if $x \geq y$ then $(x - y) \in \mathcal{P}$, then there exists an eigenvector in the cone. Moreover, if M is $u_\circ$-positive, that is, for all nontrivial x in $\mathcal{P}$ there exists scalars α, $\beta > 0$ so that $\alpha u_\circ \leq Mx \leq \beta u_\circ$, then this eigenvector is essentially unique. We also use a result from Keener and Travis [10]. Suppose operators M and N map our cone $\mathcal{P}$ back into itself, and one of them is $u_\circ$-positive. Further suppose that there exists nontrivial u, v in $\mathcal{P}$ and scalars λ, Λ so that $\lambda u \leq Mu$ and $Nv \leq \Lambda v$. Then if $Mx \leq Nx$ for all $x \in \mathcal{P}$ then $\lambda \leq \Lambda$.

In Chapter 2 we let k be a fixed element of $\{1, 2, \ldots, n-1\}$. We consider the linear differential operator $Lu = u^{(n)} + r(t)u$, where $r(t)$ is continuous on $[a, b]$. We let i_j, for $1 \leq j \leq n-k$ be integers such that $0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1$. Then our boundary conditions for this problem are given by $u^{(i)}(a) = 0$, $i = 0, 1, \ldots, k-1$ and $u^{(i_j)}(b) = 0$, $j = 1, 2, \ldots, n-k$. Now Peterson and Riddenhour [15] discovered sign conditions on the Green's function for the scalar analog of this problem. Further, they give comparisons between the Green's function for this operator with different boundary conditions. We take this eigenvalue problem and consider its corresponding integral equation. By appropriately defining a Banach space with a cone and using the sign conditions on the Green's function, we find that our integral operator in a $u_{\circ}$-positive operator. This allows us to apply the results of Krasnosel'skiǐ to get the existence of and uniqueness of an eigenvector in the cone. Further, by using the comparison between different Green's functions we can use Keener and Travis' results to give comparisons between the smallest eigenvalues of different eigenvalue problems.

In our third chapter we examin the n-point right focal eigenvalue problem $(-1)^{n-1}Lu = \lambda P(t)u$, $u^{(i-1)}(t_i) = 0$, $i = 1, 2, \ldots, n$, where $Lu = u^{(n)} + p_1(t)u^{(n-1)} + \cdots + p_n(t)$. Assuming that $Ly = 0$ is right disfocal we give an explicit form for the Green's function. Under certain sign conditions on the Green's function and conditions on $P(t)$, we can show the existence of a smallest positive eignevalue. And with further conditions on $P(t)$, that its corresponding eigenvec-

tor is essentially unique with repect to a 'cone'. We also have comparison results for the eigenvalue problem above and the problem $Lu = \Lambda Q(t)u$, $u^{(i-1)}(t_i) = 0$ for $i = 1, 2, \ldots, n$. We close this chapter by giving examples where the Green's function has the desired sign conditions.

In our final chapter we show how the results from Chapter 3 can be obtained for the n-th order linear vector difference equation $Lu(t) = \sum_{i=0}^{n} \alpha_i(t)u(t-k+i) = 0$, $t \in [a+k, b+k]$. We assume that the coefficients $\alpha_i(t)$ are defined on $[a+k, b+k]$, for $i = 1, 2, \ldots, n$, $\alpha_n(t) \equiv 1$, and $(-1)^n \alpha_o(t) > 0$ for all $t \in [a+k, b+k]$. Here, the matrix function $P(t)$ and $Q(t)$ are also defined on $[a+k, b+k]$. We assume that $Ly = 0$ is right disfocal for this difference equation case and again are able to give an explicit form for this Green's function. By assuming certain sign conditions on the Green's function we can show that the results from Chapter 3 hold for the difference eigenvalue problem. We also close this chapter by giving examples where the Green's function has the desired sign conditions.

Chapter 1

Cone Theory

I) Fundamental Definitions

We will start our exploration of cone theory with a fundamental definition. A good treatment of cone theory can be found in Deimling [2] or in Krasnosel'skiĭ [10]. Many of our definitions and theorems are from Krasnosel'skiĭ.

Definition: Let $\mathcal{B}$ be a Banach space. A nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is called a *cone* if the following conditions are satisfied:

a) The set $\mathcal{P}$ is closed;

b) If $u, v \in \mathcal{P}$, then $\alpha u + \beta v \in \mathcal{P}$ for all scalars $\alpha, \beta \geq 0$;

c) If $u, -u \in \mathcal{P}$ then $u = 0$, the zero element of $\mathcal{B}$.

We note that from b), it follows that $\mathcal{P}$ is a convex set. A cone $\mathcal{P}$ is called *solid* if it has a nonempty interior $\mathcal{P}^\circ$, that is, $\mathcal{P}^\circ \neq \emptyset$. A cone $\mathcal{P}$ is called *reproducing* if every element $x \in \mathcal{B}$ can be written in the form $x = u - v, u, v \in \mathcal{P}$. The elements u and v are not unique, for if $x = u - v,\ u, v \in \mathcal{P}$ and w is any other nonzero element of $\mathcal{P}$, then $x = (u + w) - (v + w)$.

The following lemma gives us a relationship between a solid cone and a reproducing cone.

LEMMA 1.1. *Let $\mathcal{P}$ be a solid cone in Banach space $\mathcal{B}$, then $\mathcal{P}$ is a reproducing cone.*

PROOF: Let x be an arbitrary element of $\mathcal{B}$. Let $v \in \mathcal{P}^{\circ}$, the interior of $\mathcal{P}$. Since $\mathcal{P}$ is solid, that is, $\mathcal{P}$ has nonempty interior, we know that we can find such a v. Now, since v is an interior point of $\mathcal{P}$, we have that $(v + \varepsilon x) \in \mathcal{P}$ for sufficiently small $\varepsilon > 0$. Let $u = v + \varepsilon x$ and divide through by ε to get that $\frac{1}{\varepsilon}u = \frac{1}{\varepsilon}v + x$. So we have that $x = u_{\circ} - v_{\circ}$, where $u_{\circ} = \frac{1}{\varepsilon}u$, and $v_{\circ} = \frac{1}{\varepsilon}v$, so $u_{\circ}, v_{\circ} \in \mathcal{P}$. Hence we have that $\mathcal{P}$ is a reproducing cone.

As an example, we have:

Example 1.1: Let $\mathcal{B} = C[a,b]$, the set of continuous functions on the interval $[a,b]$, with norm $||x|| = \sup_{[a,b]} |x(t)|$. Let $\mathcal{P} = C^{+}[a,b]$, the set of continuous nonnegative functions on the interval $[a,b]$. It is easy to show that $\mathcal{P}$ is a cone in $\mathcal{B}$ and has a nonempty interior $\mathcal{P}^{\circ}$, equal to the set of continuous positive functions on $[a,b]$. Then, from Lemma 1.1, we know that since $\mathcal{P}$ is solid, it is also reproducing.

The converse of Lemma 1.1 is not true. Let $\mathcal{B} = L_p[a,b]$, the space of the functions which are pth power, absolutely integrable on the interval $[a,b]$, and $\mathcal{P}$ be the set of nonnegative functions of $\mathcal{B}$. Then $\mathcal{P}$ is a cone and although $\mathcal{P}$ is reproducing, it has no interior points. It can be shown that in a finite dimensional space, solid and reproducing are equivalent.

The space $\mathcal{B}$ is called *partially ordered* if, for certain elements $x, y \in \mathcal{B}$, the

relationship $x \leq y$ is defined and the relation sign'$\leq$' posseses the properties:

i) If $x \leq y$, then $\alpha x \leq \alpha y$ for all scalars $\alpha \geq 0$, and $\alpha x \geq \alpha y$ if $\alpha < 0$;

ii) If $x \leq y$ and $y \leq x$ then $x = y$;

iii) If $x_1 \leq y_1$ and $x_2 \leq y_2$ then $(x_1 + x_2) \leq (y_1 + y_2)$;

iv) If $x \leq y$ and $y \leq z$ then $x \leq z$.

Our cone $\mathcal{P}$ induces a partial ordering on $\mathcal{B}$ if we write $x \leq y$ to mean that $(y - x) \in \mathcal{P}$. To see this, we will show that this relation satisfies the properties listed above. Property i) follows from property b) of a cone, for if $x \leq y$, then $(y - x) \in \mathcal{P}$ from which we have that $\alpha(y - x) \in \mathcal{P}$ for all $\alpha \geq 0$. So $\alpha(y - x) = (\alpha y - \alpha x) \in \mathcal{P}$, or that $\alpha x \leq \alpha y$ for all $\alpha \geq 0$. If $\alpha < 0$, then $(-\alpha)(y - x) \in \mathcal{P}$ or $(\alpha x - \alpha y) \in \mathcal{P}$, so that $\alpha y \leq \alpha x$ for all $\alpha < 0$.

To show ii), we note that if $x \leq y$ and $y \leq x$ then $(y - x), (x - y) \in \mathcal{P}$. But $(y - x) = -(x - y)$ so we have that $-(x - y), (x - y) \in \mathcal{P}$ which implies from property c) of a cone, that $(x - y) = 0$ or $x = y$.

The last two properties follow from the fact that the cone is closed under addition. If $x_1 \leq y_1$ and $x_2 \leq y_2$ then $(y_1 - x_1), (y_2 - x_2) \in \mathcal{P}$. So then $(y_1 - x_1) + (y_2 - x_2) \in \mathcal{P}$. But $(y_1 - x_1) + (y_2 - x_2) = (y_1 + y_2) - (x_1 + x_2)$ so we have that $(x_1 + x_2) \leq (y_1 + y_2)$. Finally, if $x \leq y$ and $y \leq z$ then $(y - x), (z - y) \in \mathcal{P}$. So $(y - x) + (z - y) = (z - x) \in \mathcal{P}$, and $x \leq z$ follows.

One further property of the '$\leq$' relation is invoked by the fact that the cone is closed. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{B}$ and that $x_n \to x$ and

$y_n \to y$ as $n \to \infty$. Suppose further that $x_n \leq y_n$ for $n = 1, 2, \ldots$, then $x \leq y$. The proof is simple. If $x_n \leq y_n$ for $n = 1, 2, \ldots$, then $(y_n - x_n) \in \mathcal{P}$ for $n = 1, 2, \ldots$. Now $(y_n - x_n) \to (y - x)$ as $n \to \infty$ so we have that $(y - x)$ is the limit point of the sequence $\{y_n - x_n\}$. Thus, since $\{y_n - x_n\} \subset \mathcal{P}$ and $\mathcal{P}$ is closed (hence it contains all of its limit points), we have that $(y - x) \in \mathcal{P}$ or $x \leq y$.

II) Preliminary Lemmas

Consider the cone in $\mathcal{R}^2$, $\mathcal{P} = \{(r, \theta) \,|\, r \geq 0, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}\}$, where the point (r, θ) is given in polar coordinates. We notice that no line lies completely in the cone. A line segment may lie in $\mathcal{P}$ or at best a ray will lie in the cone. This geometric property also holds true for abstract cones, as the following lemmas from Krasnosel'skiǐ demonstrate.

LEMMA 1.2. *Let $u_\circ \in \mathcal{P}$ and $x \in \mathcal{B}$. Suppose there exists an $\alpha_\circ$ so that $x \leq \alpha_\circ u_\circ$. Then $x \leq \alpha u_\circ$ for all $\alpha \geq \alpha_\circ$.*

PROOF: Let α be greater than or equal to $\alpha_\circ$. Let $\beta = (\alpha - \alpha_\circ) \geq 0$. Then since $(\alpha_\circ u - x), \beta u_\circ \in \mathcal{P}$, we have that $(\alpha u_\circ - x) = (\alpha_\circ + \beta) u_\circ - x = (\alpha_\circ u_\circ - x) + \beta u_\circ \in \mathcal{P}$. So we have that $x \leq \alpha u_\circ$.

LEMMA 1.3. *Let $u_\circ \in \mathcal{P} \backslash \{0\}$ and $x \in \mathcal{B}$. Suppose there exists an $\alpha_\circ$ so that $x \leq \alpha_\circ u_\circ$. Then there exists a smallest α_1 for which $x \leq \alpha_1 u_\circ$.*

PROOF: First, suppose there does not exist a lower bound on the set of all α's for which $x \leq \alpha u_\circ$. Then we can find a negative sequence $\{\beta_n\}$ of this set, where

$\beta_n \to -\infty$ an $n \to \infty$. Then for $n = 1, 2, \ldots$, we have that $x \leq \beta_n u_\circ$. Now $|\beta_n| > 0$ for all n so we have that $\frac{1}{|\beta_n|}x \leq \frac{\beta_n}{|\beta_n|}u_\circ = -u_\circ$. Letting $n \to \infty$ we find that $0 \leq -u_\circ$, which tells us that $-u_\circ \in \mathcal{P}$ which contradicts $u_\circ \in \mathcal{P}\backslash\{0\}$. Thus there exists a lower bound for this set. Let $\alpha_1$be the greatest lower bound of this set. Then $\alpha_1 u_\circ - x$ is a limit point of our cone, and since our cone is closed we have the desired result.

LEMMA 1.4. *Let $x, u_\circ \in \mathcal{B}$ and $-u_\circ \notin \mathcal{P}$. Suppose there exists an $\alpha_\circ$ so that $\alpha_\circ u_\circ \leq x$. Then there exists a maximum α_1 so that $\alpha_1 u_\circ \leq x$.*

PROOF: Suppose there does not exists an upper bound on the set of all α's such that $\alpha u_\circ \leq x$. Then we can find a positive sequence $\{\beta_n\}$ of this set, where $\beta_n \to \infty$ as $n \to \infty$. This gives us that for $n = 1, 2, \ldots$, $u_\circ \leq \frac{1}{\beta_n}x$. Letting $n \to \infty$ yields that $u_\circ \leq 0$ or $-u_\circ \in \mathcal{P}$ which is a contradiction. Thus there exists an upper bound of this set. Let α_1 be the least upper bound of this set. Then we have that $x - \alpha_1 u_\circ$ is a limit point of our cone, and since our cone is closed we have that $\alpha_1 u_\circ \leq x$.

III) Linear Positive Operators

In this section we study linear operators which leave a cone invariant in a Banach space. Under some general assumptions, these operators will have a characteristic vector in the cone. Before getting on to these theorems, we will need some additional definitions.

Let $\mathcal{P}$ be a cone in a Banach space $\mathcal{B}$. The operator $M : \mathcal{B} \to \mathcal{B}$ is called

positive if M maps the cone back into itself, that is, $M : \mathcal{P} \to \mathcal{P}$. If $u_\circ$ is a nonzero element of $\mathcal{P}$, then we call the linear operator M *$u_\circ$-bounded below* if for every nonzero $x \in \mathcal{P}$ there is a natural number n and an $\alpha > 0$ (which may depend on x) such that $\alpha u_\circ \leq M^n x$. {where $M^n x$ means the operator M operating on the element x, n-times.} An operator M is called *$u_\circ$-bounded above* if for every nonzero $x \in \mathcal{P}$ there is a natural number n and a $\beta > 0$ (which may depend on x) such that $M^n x \leq \beta u_\circ$. If for every nonzero $x \in \mathcal{P}$ there exists a natural number n and $\alpha, \beta > 0$ such that $\alpha u_\circ \leq M^n x \leq \beta u_\circ$, then we say that M is a *$u_\circ$-positive* operator.

A property of M being $u_\circ$-bounded below is that if $x \in \mathcal{P}\backslash\{0\}$, then $M^n x \neq 0$ for any n. For suppose that there existed an $x \in \mathcal{P}\backslash\{0\}$ and an integer k such that $M^k x = 0$. Let us suppose further that k is the smallest positive integer for which this holds. Then $M^{k-1}x \in \mathcal{P}\backslash\{0\}$ and since M is $u_\circ$-bounded below, there exists an n and an $\alpha > 0$ so that $\alpha u_\circ \leq M^n(M^{k-1}x)$. But $M^n(M^{k-1}x) = M^{n-1}(M^k x) = M^{n-1}(0) = 0$. Hence $\alpha u_\circ \leq 0$ or $-\alpha u_\circ \in \mathcal{P}$. Since $\alpha > 0$ this tells us that $-u_\circ \in \mathcal{P}$ which is a contradiction and our claim is proved.

In the above definitions, we have found a natural number n, and then operated on x with M, n times. In our study of differential equations, we will always take this natural number n to be identically equal to 1. So, for example, we would say that M is *$u_\circ$-positive* if for every nonzero $x \in \mathcal{P}$ there exists $\alpha, \beta > 0$ such that $\alpha u_\circ \leq Mx \leq \beta u_\circ$. We will, however, continue to keep these more general

definitions a while longer.

If our cone $\mathcal{P}$ is solid and if for ever nonzero $x \in \mathcal{P}$ we can find an n such that $M^n x$ is in the interior of $\mathcal{P}$, then we say that M is *strongly positive*. A strongly positive operator is the simpliest example of a $u_\circ$-positive operator. This is seen in the following Lemma.

LEMMA 1.5. *Let M be a strongly positive linear operator, relative to the solid cone $\mathcal{P}$. Then for any $u_\circ$ in the interior of $\mathcal{P}$, M is $u_\circ$-positive.*

PROOF: Let $u_\circ \in \mathcal{P}^\circ$ and x an arbitrary, nonzero element of $\mathcal{P}$. Then, since M is a strongly positive operator, there exists a natural number n such that $M^n x \in \mathcal{P}^\circ$. Since $M^n x$ is an interior point of $\mathcal{P}$, we have that $(M^n x - \alpha u_\circ) \in \mathcal{P}$ for sufficiently small $\alpha > 0$, that is $\alpha u_\circ \leq M^n x$. Similarly, since $u_\circ$ is an interior point of $\mathcal{P}$, we have that $(u_\circ - \frac{1}{\beta} M^n x) \in \mathcal{P}$, for sufficiently large $\beta > 0$. So $\frac{1}{\beta} M^n \leq u_\circ$, which gives us that $M^n x \leq \beta u_\circ$. So for an arbitrary nonzero x in $\mathcal{P}$, we have found an $\alpha, \beta > 0$ and an n, so that $\alpha u_\circ \leq M^n x \leq \beta u_\circ$.

IV) Characteristic Vectors

Let $\mathcal{B}$ be a Banach space and M an operator on $\mathcal{B}$. We call a nonzero element $x \in \mathcal{B}$ an *eigenvector* or *characteristic vector* if there exists a scalar λ, such that $Mx = \lambda x$. The scalar λ is called an *eigenvalue* or a *characteristic value*. We sometimes call (λ, x) an *eigenpair* for the operator M.

Suppose that M is a linear operator which leaves some cone $\mathcal{P} \subset \mathcal{B}$ invariant, that is, M is a positive operator with respect to $\mathcal{P}$. If x is an eigenvector of M

and x is an element of $\mathcal{P}$, then we say that x is a *positive* eigenvector, and its associated eigenvalue is called a *positive* eigenvalue. We are interested in these positive eigenvectors and will employ *Schauder's Theorem* to prove their existence.

SCHAUDER'S FIXED POINT THEOREM. *Let M be a completely continuous operator which maps a closed convex bounded set $\mathcal{K}$, into itself. Then M has a fixed point $x_\circ \in \mathcal{K}$ which satisfies the equation $Mx = x$.*

Recall that an operator is called *completely continuous* if it is continuous and maps bounded sets into sets whose closures are compact. We sometimes will refer to a completely continuous operator as a *compact* operator.

The following Theorem from Krasnosel'skiĭ [10], gives conditions for a linear, completely continuous operator to possess a positive eigenvector.

THEOREM 1.6. *Let $\mathcal{B}$ be a Banach space with cone $\mathcal{P}$, and M a completely continuous, positive linear operator. Suppose there exists an $\alpha > 0$, a natural number p, and a $u \in \mathcal{B}$, $-u \notin \mathcal{P}$, $u = v - w$, $v, w \in \mathcal{P}$, such that $M^p u \geq \alpha u$. Then M has a eigenvector $x_\circ \in \mathcal{P}$ and its associated eigenvalue $\lambda_\circ$, satisfies the inequality $\lambda_\circ \geq \sqrt[p]{\alpha}$.*

PROOF: Let K be the intersection of the closed unit ball with the cone, so $K = P \cap \{x \in \mathcal{B} : ||x|| \leq 1\}$. We have that $u = (v - w)$, $v, w \in \mathcal{P}$ and $-u \notin \mathcal{P}$. So we know that $v \neq 0$. We then define the operators M_n on K by

$$(1) \qquad M_n x = \frac{M(x + \frac{v}{n})}{||M(x + \frac{v}{n})||}, \qquad n = 1, 2, \ldots$$

Then, for each n, M_n is completely continuous since the operator M is completely continuous. We also have that, for each n, $M_n : K \to K$. This is easily seen, for if x is an arbitrary element of K, then $(x + \frac{v}{n}) \in \mathcal{P}$ and since M is a positive operator, $M(x + \frac{v}{n}) \in \mathcal{P}$ so that $M_n x \in \mathcal{P}$. Also, by (1) it is clear that $||M_n x|| \leq 1$, so we have that $M_n x \in K$. Since x was arbitrary we have that $M_n : K \to K$.

Now K is a closed bounded convex set. {We know that K is convex since both $\mathcal{P}$ and the unit ball are convex and the intersection of two convex sets is a convex set.} By applying Schauder's Theorem we have that every operator M_n has a fixed point x_n in K. So we have that $M_n x_n = x_n$ or from (1) we have that

$$(2) \qquad M(x_n + \frac{v}{n}) = \lambda_n x_n, \quad \lambda_n = ||M(x_n + \frac{v}{n})||, \quad n = 1, 2, \ldots .$$

Now M is a compact operator and the set $\{x_n + \frac{v}{n} \,|\, n = 1, 2, \ldots\}$ is bounded by $1 + ||v||$, since $||x_n + \frac{v}{n}|| \leq ||x_n|| + \frac{||v||}{n} \leq 1 + ||v||$. Thus there exists a subsequence of the sequence $\{M(x_n + \frac{v}{n})\}$ which converges. That is, there exists a sequence of n_i's, so that the sequence from (2), $\{\lambda_{n_i} x_{n_i}\}$ converges. Now, since the sequence $\{M(x_{n_i} + \frac{v}{n_i})\}$ converges, this gives us that $\lambda_{n_i} = ||M(x_{n_i} + \frac{v}{n_i})||$, converges to, say, $\lambda_o \geq 0$.

We will now show that $\lambda_o > 0$ and, in fact that $\lambda_o \geq \sqrt[p]{\alpha}$. From (2) we have that $M(x_n + \frac{v}{n}) = \lambda_n x_n$. Thus $\lambda_n x_n \geq M x_n$, since M is linear and $(\lambda_n x_n - M x_n) = \frac{1}{n} M v \in \mathcal{P}$. This gives us that $x_n \geq \frac{1}{\lambda_n} M x_n$. Now since M is a positive linear operator we have that $x_n \geq \frac{1}{\lambda_n} M x_n \geq \frac{1}{\lambda_n} M[\frac{1}{\lambda_n} M x_n] = \frac{1}{\lambda_n^2} M^2 x_n$. So, after $p - 1$ iterations we get that $x_n \geq \frac{1}{\lambda_n^{p-1}} M^{p-1} x_n$, so from (2) we have $x_n \geq$

$\frac{1}{\lambda_n^p} M^p(x_n + \frac{v}{n})$.

Now $M^p(x_n + \frac{v}{n}) \geq M^p(\frac{v}{n}) = \frac{1}{n} M^p v$, since $(x_n + \frac{v}{n}) \geq \frac{v}{n}$. Also we have that $M^p v \geq M^p u$ since $v \geq u = v - w$, and $M^p u \geq \alpha u$ by hypothesis. This gives us that

$$\begin{aligned} x_n &\geq \frac{1}{\lambda_n^p} M^p(x_n + \frac{v}{n}) \\ &\geq \frac{1}{n\lambda_n^p} M^p v \\ &\geq \frac{\alpha}{n\lambda_n^p} u, \qquad \text{for } n = 1, 2, \ldots. \end{aligned}$$

By Lemma 1.4 there exits a sequence of maximum β_n's so that

$$x_n \geq \beta_n u, \qquad n = 1, 2, \ldots. \tag{3}$$

This and the fact that $v \geq u$ yields

$$\begin{aligned} x_n &\geq \frac{1}{\lambda_n^p} M^p(x_n + \frac{v}{n}) \\ &\geq \frac{1}{\lambda_n^p} M^p(\beta_n u + \frac{u}{n}) \\ &= \frac{1}{\lambda_n^p}(\beta_n + \frac{1}{n}) M^p u \\ &\geq \frac{\alpha}{\lambda_n^p}(\beta_n + \frac{1}{n}) u, \qquad n = 1, 2, \ldots \end{aligned} \tag{4}$$

Now, by the maximality of each β_n, we combine (3) and (4) to get that

$$\begin{aligned} \beta_n &\geq \frac{\alpha}{\lambda_n^p}(\beta_n + \frac{1}{n}) \\ \text{or} \quad \lambda_n^p &\geq \alpha + \frac{\alpha}{n\beta_n}, \qquad n = 1, 2, \ldots. \end{aligned}$$

Now $\lambda_{n_i} \to \lambda_o$ as $i \to \infty$, hence from the last inequality we get that $\lambda_o^p \geq \alpha > 0$ or our desired inequality, $\lambda_o \geq \sqrt[p]{\alpha}$.

Since we have that the sequence $\{\lambda_{n_i} x_{n_i}\}$ converges, and λ_{n_i} converges to $\lambda_o \geq \sqrt[p]{\alpha} > 0$, we have that the sequence $\{\lambda_{n_i} x_{n_i}\}$ converges to $\lambda_o x_o$ for some x_o. Hence, we have that $\{(\lambda_{n_i} x_{n_i})/\lambda_o\}$ converges to x_o as $i \to \infty$. Now from (2) we have that $\|(\lambda_{n_i} x_{n_i})/\lambda_o\| = (\frac{\lambda_{n_i}}{\lambda_o})\|x_{n_i}\| = \frac{\lambda_{n_i}}{\lambda_o} \to 1$ as $i \to \infty$. Thus $\|x_o\| = 1$ and so $x_o \neq 0$.

Since each $(x_{n_i} + \frac{v}{n_i}) \in \mathcal{P}$ and $\mathcal{P}$ is closed, we have that $x_o \in \mathcal{P}$. Also, since M is continuous, we have that $M(x_{n_i} + \frac{v}{n_i})$ converges to Mx_o. But $M(x_{n_i} + \frac{v}{n_i}) = \lambda_{n_i} x_{n_i}$ which converges to $\lambda_o x_o$. Thus $Mx_o = \lambda_o x_o$. That is, x_o is an eigenvector of the operator M with eigenvalue λ_o, and further, $x_o \in \mathcal{P}$. Hence our theorem is proved.

An eigenvalue, λ_o, of an operator M is sometimes called *simple* if all the solutions of the equation

$$(M - \lambda_o I)^n x = 0, \qquad n = 1, 2, \ldots$$

are also solutions of $(M - \lambda_o I)x = 0$, where this set of solutions is one dimensional. Krasnosel'skiĭ has an important result which gives conditions under which our positive eigenvalue is simple and its corresponding eigenvector is essentially unique. To prove this theorem, we will first need the following lemma.

LEMMA 1.7. *Let x_o be a positive eigenvector of the u_o-positive operator M. Then M is an x_o-positive operator.*

PROOF: Since $x_\circ \in \mathcal{P}$ and M is $u_\circ$-positive, there exists as $\alpha_\circ, \beta_\circ > 0$ and a natural number p so that

$$\alpha_\circ u_\circ \leq M^p x_\circ \leq \beta_\circ u_\circ.$$

Now if $\lambda_\circ$ is the corresponding eigenvalue of $x_\circ$, then since M is $u_\circ$-positive, we know that $\lambda_\circ > 0$. Further we have that $M^p x_\circ = \lambda_\circ^p x_\circ$, and so

$$(5) \qquad \frac{\lambda_\circ^p}{\beta_\circ} x_\circ \leq u_\circ \quad \text{and} \quad u_\circ \leq \frac{\lambda_\circ^p}{\alpha_\circ} x_\circ.$$

So if x is an arbitary element of $\mathcal{P}\backslash\{0\}$, then there exists an $\alpha, \beta > 0$ and a natural number n so that

$$\alpha u_\circ \leq M^n x \leq \beta u_\circ.$$

But by the inequalities (5) we have

$$\alpha_1 x_\circ \leq M^n x \leq \beta_1 x_\circ,$$

where $\alpha_1 = \alpha\lambda_\circ^p/\beta_\circ$ and $\beta_1 = \beta\lambda_\circ^p/\alpha_\circ$. Thus M is $x_\circ$-positive.

With this lemma proved, we move onto our important theorem.

THEOREM 1.8. *Let $\mathcal{P}$ be a reproducing cone in our Banach Space $\mathcal{B}$, and M a completely continuous, $u_\circ$-positive linear operator on $\mathcal{B}$. Then the operator M has an essentially unique, (unique to within the norm), eigenvector in $\mathcal{P}$, and its associated eigenvalue is simple.*

PROOF: Since M is $u_\circ$-positive, and $u_\circ \in \mathcal{P}$, there exists $\alpha, \beta > 0$ such that $\alpha u_\circ \leq M^p u_\circ \leq \beta u_\circ$, for some natural number p. Now $\mathcal{P}$ is a reproducing cone,

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so $u_\circ = (v - w)$, $v, w \in \mathcal{P}$ and $-u_\circ \notin \mathcal{P}$. Thus, by Theorem 1.6, M has an eigenvector $x_\circ$, in $\mathcal{P}$ with associated eigenvalue $\lambda_\circ$, and since M is $u_\circ$-positive, we know that $\lambda_\circ > 0$.

We will first show that $\lambda_\circ$ is simple, and then show that there is no other eigenvalue with corresponding eigenvector in $\mathcal{P}$. First suppose that there exists a $y_\circ$, noncolinear to $x_\circ$, so that $My_\circ = \lambda_\circ y_\circ$. We can assume that $-y_\circ \notin \mathcal{P}$, for if it were, we can take $z_\circ = -y_\circ$, so $-z_\circ \notin \mathcal{P}$ and $Mz_\circ = \lambda_\circ z_\circ$. Since $\mathcal{P}$ is reproducing, there exists $y_1, y_2 \in \mathcal{P}$ so that $y_\circ = y_1 - y_2$ with $y_1 \neq 0$ since $-y_\circ \notin \mathcal{P}$. We note that $y_\circ \leq y_1$ since $(y_1 - y_\circ) = y_2 \in \mathcal{P}$.

Now from Lemma 1.7 we have that M is $x_\circ$-positive. Hence there exists a $\beta > 0$ and a p so that $M^p y_1 \leq \beta x_\circ$. This gives us that $\lambda_\circ^p y_\circ = M^p y_\circ \leq M^p y_1 \leq \beta x_\circ$. Thus we have that $(x_\circ - \frac{\lambda_\circ^p}{\beta} y_\circ) \in \mathcal{P}$. Then, by Lemma 1.4, there exists a maximal $\beta_\circ$ so that $(x_\circ - \beta_\circ y_\circ) \in \mathcal{P}$, which gives us that $\beta_\circ \geq \frac{\lambda_\circ^p}{\beta} > 0$.

We have that $(x_\circ - \beta_\circ y_\circ) \in \mathcal{P}$, and M is $x_\circ$-positive, thus there exists an $\alpha > 0$, and an n such that $\alpha x_\circ \leq M^n(x_\circ - \beta_\circ y_\circ)$. Now $\beta_\circ y_\circ \leq x_\circ$, so $\alpha\beta_\circ y_\circ \leq \alpha x_\circ$. But this give that $\alpha\beta_\circ y_\circ \leq \alpha x_\circ \leq M^n(x_\circ - \beta_\circ y_\circ)$, or that $\alpha\beta_\circ y_\circ \leq (M^n x_\circ - \beta_\circ M^n y_\circ) = \lambda_\circ^n x_\circ - \beta_\circ \lambda_\circ^n y_\circ$. Thus $\frac{\alpha\beta_\circ}{\lambda_\circ^n} y_\circ \leq x_\circ - \beta_\circ y_\circ$. But this gives us that $(x_\circ - \beta_\circ(1 + \frac{\alpha}{\lambda_\circ^n}) y_\circ) \in \mathcal{P}$ which contradicts the maximality of $\beta_\circ$. Thus the only solutions to $(M - \lambda_\circ I)x = 0$ are scalar multiples of $x_\circ$.

Next we suppose that there exists a $n_\circ$ and a $z_\circ$, noncolinear to $x_\circ$, such that $(M - \lambda_\circ I)^{n_\circ} z_\circ = 0$, and $(M - \lambda_\circ I)^{n_\circ - 1} z_\circ \neq 0$. Let $z_1 = (M - \lambda_\circ I)^{n_\circ - 1} z_\circ$, so that

$(M - \lambda_\circ I)z_1 = 0$. From what we have shown earlier we have that $z_1 = kx_\circ$, $k \neq 0$. This gives us that $\frac{1}{k}(M - \lambda_\circ I)^{n_\circ - 1} z_\circ = x_\circ$. Now let $z_2 = -\frac{1}{k}(M - \lambda_\circ I)^{n_\circ - 2} z_\circ$, so that $(M - \lambda_\circ I)z_2 = -x_\circ$ or $Mz_2 = \lambda_\circ z_2 - x_\circ$. Now M is linear so

$$\begin{aligned} M^2 z_2 &= M(\lambda_\circ z_2 - x_\circ) \\ &= \lambda_\circ M z_2 - M x_\circ \\ &= \lambda_\circ(\lambda_\circ z_2 - x_\circ) - \lambda_\circ x_\circ \\ &= \lambda_\circ^2 z_2 - 2\lambda_\circ x_\circ. \end{aligned}$$

By induction we have that

$$M^n z_2 = \lambda_\circ^n z_2 - n\lambda_\circ^{n-1} x_\circ. \tag{6}$$

Now $z_2 \notin \mathcal{P}$, for if it were, then $M^n z_2 \in \mathcal{P}$. Then from (6), after we divide through by $(n\lambda_\circ^{n-1})$, we get that $(\frac{\lambda_\circ}{n} z_2 - x_\circ) \in \mathcal{P}$. But our cone is closed and $-x_\circ$ is a limit point of $(\frac{\lambda_\circ}{n} z_2 - x_\circ)$, so we would have that $-x_\circ \in \mathcal{P}$. But $x_\circ$ is an eigenvector of $\mathcal{P}$, so $x_\circ \neq 0$ and this contradicts item c) in our definition of a cone.

Our cone $\mathcal{P}$ is reproducing so $z_2 = (v - w)$, $v, w \in \mathcal{P}$, and $w \neq 0$ since $z_2 \notin \mathcal{P}$. Now $-w \leq z_2$ since $(z_2 + w) = v \in \mathcal{P}$. Since $w \in \mathcal{P}\backslash\{0\}$ and M is $x_\circ$-positive, there exists a $\beta > 0$ and a p such that $M^p w \leq \beta x_\circ$. This gives us that

$$\begin{aligned} -\beta x_\circ &\leq M^p(-w) \\ &\leq M^p z_2 = \lambda_\circ^p z_2 - p\lambda_\circ^{p-1} x_\circ \qquad \text{from (6)} \end{aligned}$$

$$\text{or} \quad z_2 \geq \frac{p\lambda_\circ^{p-1} - \beta}{\lambda_\circ^p} x_\circ,$$

where $(p\lambda_\circ^{p-1} - \beta) < 0$ since $z_2 \notin \mathcal{P}$. Then $-z_2 \leq \frac{\beta - p\lambda_\circ^{p-1}}{\lambda_\circ^p} x_\circ$. Multiplying through by the positive quantity $\frac{\lambda_\circ^p}{\beta - p\lambda_\circ^{p-1}}$, we get that $(x_\circ + (\frac{\lambda_\circ^p}{\beta - p\lambda_\circ^{p-1}})z_2) \in \mathcal{P}$. So by Lemma 1.4, there exists a maximal $\beta_\circ > 0$, so that $(x_\circ + \beta_\circ z_2) \in \mathcal{P}$. This gives us that $M(x_\circ + \beta_\circ z_2) \in \mathcal{P}$. But

$$\begin{aligned} M(x_\circ + \beta_\circ z_2) &= Mx_\circ + \beta_\circ M z_2 \\ &= \lambda_\circ x_\circ + \beta_\circ(\lambda_\circ z_2 - x_\circ) \\ &= (\lambda_\circ - \beta_\circ)x_\circ + \beta_\circ \lambda_\circ z_2. \end{aligned}$$

Now $(\lambda_\circ - \beta_\circ) > 0$ for if $(\lambda_\circ - \beta_\circ) < 0$, then $([-(\lambda_\circ - \beta_\circ)]x_\circ) \in \mathcal{P}$ and since $((\lambda_\circ - \beta_\circ)x_\circ + \beta_\circ\lambda_\circ z_2) \in \mathcal{P}$ then their sum is in $\mathcal{P}$, that is $(\beta_\circ\lambda_\circ z_2) \in \mathcal{P}$, which contradicts $z_2 \notin \mathcal{P}$. Thus $(x_\circ + \frac{\beta_\circ\lambda_\circ}{\lambda_\circ - \beta_\circ} z_2) \in \mathcal{P}$. But $\frac{\beta_\circ\lambda_\circ}{\lambda_\circ - \beta_\circ} > \beta_\circ$. This contradicts the maximality of $\beta_\circ$.

Thus the solutions of the equation

$$(M - \lambda_\circ I)^n x = 0, \qquad n = 1, 2, \ldots$$

cannot be different from the solutions to

$$(M - \lambda_\circ I)x = 0.$$

Hence we have that $\lambda_\circ$ is a simple positive eigenvalue.

We now prove the second half of our theorem, that the eigenvectors of M in $\mathcal{P}$ are essentially unique. Let us assume that there exists two linearly independent

eigenvectors $x_1, x_2 \in \mathcal{P}$ such that $Mx_1 = \lambda_1 x_1$ and $Mx_2 = \lambda_2 x_2$ and that $||x_1|| = ||x_2|| = 1$. Then by the first half of our theorem, $\lambda_1 \neq \lambda_2$, so we can assume that $\lambda_1 > \lambda_2$. Since M is $u_\circ$-positive, we know that $\lambda_2 > 0$.

Now by Lemma 1.7, M is x_1-positive, so there exists an $\alpha > 0$ and a p so that $M^p x_2 = \lambda_2^p x_2 \geq \alpha x_1$. This gives us that $(\lambda_2^p x_2 - \alpha x_1) \in \mathcal{P}$ and since $\lambda_2^p > 0$ we have that $(x_2 - \frac{\alpha}{\lambda_2^p} x_1) \in \mathcal{P}$. So, by Lemma 1.4, there exists a maximal $\beta_\circ$ so that $(x_2 - \beta_\circ x_1) \in \mathcal{P}$. Thus $M(x_2 - \beta_\circ x_1)$ is an element of $\mathcal{P}$. But $M(x_2 - \beta_\circ x_1) = Mx_2 - \beta_\circ M x_1 = \lambda_2 x_2 - \beta_\circ \lambda_1 x_1$. Hence we have that $(x_2 - \beta_\circ \frac{\lambda_1}{\lambda_2} x_1) \in \mathcal{P}$. So by the maximality of $\beta_\circ$ we must have that $\frac{\lambda_1}{\lambda_2} \leq 1$, or $\lambda_1 \leq \lambda_2$. But this contradicts our assumption that $\lambda_1 > \lambda_2$. Hence the eigenvector of M in $\mathcal{P}$ is essentially unique, and the proof of our theorem is complete.

The next two theorems, both from Krasnosel'skiǐ, we state without proof.

THEOREM 1.9. *Let $\mathcal{P}$ be a reproducing cone in Banach space $\mathcal{B}$, and M a completely continuous, $u_\circ$-positive linear operator on $\mathcal{B}$. Then the eigenvalue corresponding to the essentially unique positive eigenvector in $\mathcal{P}$, is greater that the absolute magnitudes of the remaining eigenvalues.*

THEOREM 1.10. *Let $x_\circ$ be a positive eigenvector, with corresponding eigenvalue $\lambda_\circ$, of the completely continuous $x_\circ$-bounded above linear operator M. If the cone $\mathcal{P}$ is reproducing, then the remaining eigenvalues of the operator M are, in modulus, not greater than $\lambda_\circ$.*

As mentioned before, in the study of differential equations, one often takes the definition of a $u_\circ$-positive operator to be slightly different. For our purposes we will call an operator M $u_\circ$-positive if for every $x \in \mathcal{P}\backslash\{0\}$, there exists an $\alpha, \beta > 0$ so that $\alpha u_\circ \leq Mx \leq \beta u_\circ$. From this point on, this will be the definition we will be using.

For the final theorem in this chapter, we will be needing one more definition. If M and N are two linear operators which map $\mathcal{B}$ back into itself, then we say that $M \leq N$ (with respect to $\mathcal{P}$), if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

This last result was discovered by Keener and Travis [8,9,15]. It gives comparison results between two eigenvalues of two different operators.

THEOREM 1.11. *Let $\mathcal{P}$ be a cone in the Banach space $\mathcal{B}$. Let $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators, one of which is $u_\circ$-positive. If $M \leq N$ and there exists nontrivial $u_1, u_2 \in \mathcal{P}$ and $\lambda_1, \lambda_2 > 0$ such that*

$$\lambda_1 u_1 \leq Mu_1 \qquad \text{and} \qquad Nu_2 \leq \lambda_2 u_2,$$

then $\lambda_1 \leq \lambda_2$. Moreover, if $\lambda_1 = \lambda_2$ then u_1 is a scalar multiple of u_2.

PROOF: We will first assume that M is a $u_\circ$-positive operator. Then, since $u_1 \in \mathcal{P}\backslash\{0\}$, there exists a $\beta_1 > 0$ so that $Mu_1 \leq \beta_1 u_\circ$. But $\lambda_1 u_1 \leq Mu_1$, so we have that $\lambda_1 u_1 \leq \beta_1 u_\circ$ or $\frac{\lambda_1}{\beta_1} u_1 \leq u_\circ$.

Now $u_2 \in \mathcal{P}\backslash\{0\}$ so there exists an $\alpha_2 > 0$ so that $\alpha_2 u_\circ \leq Mu_2$. But $M \leq N$, so $\alpha_2 u_\circ \leq Mu_2 \leq Nu_2$. This gives us that $\alpha_2(\frac{\lambda_1}{\beta_1})u_1 \leq \alpha_2 u_\circ \leq Nu_2 \leq \lambda_2 u_2$. So $\frac{\alpha_2 \lambda_1}{\beta_1} v_1 \leq \lambda_2 u_2$. Thus $(u_2 - \beta u_1) \in \mathcal{P}$, where $\beta = \frac{\alpha_2 \lambda_1}{\beta_1 \lambda_2} > 0$.

By Lemma 1.4, there exists a maximal $\beta_\circ$ so that $(u_2 - \beta_\circ u_1) \in \mathcal{P}$. Now

$$\begin{aligned} M(u_2 - \beta_\circ u_1) &= Mu_2 - M(\beta_\circ u_1) \\ &\leq Nu_2 - \beta_\circ M u_1 \\ &\leq \lambda_2 u_2 - \beta_\circ \lambda_1 u_1. \end{aligned}$$

Thus $(u_2 - \beta_\circ(\frac{\lambda_1}{\lambda_2})u_1) \in \mathcal{P}$. So by the maximality of $\beta_\circ$ we must have that $\frac{\lambda_1}{\lambda_2} \leq 1$ or $\lambda_1 \leq \lambda_2$.

Finally, suppose that $\lambda_1 = \lambda_2 \doteq \lambda$. From above we have that $(u_2 - \beta_\circ u_1) \in \mathcal{P}$, where $\beta_\circ$ is the maximal scalar given to us by Lemma 1.4. If $(u_2 - \beta_\circ u_1) = 0$, then we are done. If not, then there exists an $\alpha > 0$ so that $\alpha u_\circ \leq M(u_2 - \beta_\circ u_1)$. Now from above, we have that $\frac{\lambda}{\beta_1} u_1 \leq u_\circ$ so that

$$\begin{aligned} \alpha \frac{\lambda}{\beta_1} u_1 &\leq \alpha u_\circ \\ &\leq M(u_2 - \beta_\circ u_1) \\ &= Mu_2 - \beta_\circ M u_1 \\ &\leq Nu_2 - \beta_\circ M u_1 \\ &\leq \lambda u_2 - \beta_\circ \lambda u_1 \end{aligned}$$

Thus $(u_2 - \beta_\circ u_1 - \frac{\alpha}{\beta_1} u_1) = (u_2 - (\beta_\circ + \frac{\alpha}{\beta_1})u_1) \in \mathcal{P}$. But $\frac{\alpha}{\beta_1} > 0$ contradicts the maximality of $\beta_\circ$. Thus we must have that $u_2 = \beta_\circ u_1$.

When we started we assumed that M was the $u_\circ$-positive operator. If N is the $u_\circ$-positive operator, then the proof is very similar, and so will not be repeated.

Chapter 2

Comparison Theorems for a Right Disfocal Eigenvalue Problem

I) <u>INTRODUCTION</u>:

Let $n > 1$, $m \geq 1$, k a fixed element of $\{1, 2, \ldots, n-1\}$ and $I = [a, b]$. We define the linear differential operator L, by $Lu = u^{(n)} + r(t)u$, where $u(t)$ is an m-column vector function, and $r(t)$ is a continuous function on $[a, b]$. Also let $P(t) = (p_{ij}(t))$, $Q(t) = (q_{ij}(t))$ be continuous $m \times m$ matrix functions on $[a, b]$ and let i_j, for $1 \leq j \leq n-k$ be integers such that $0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1$.

We consider the two point right focal eigenvalue problem:

(1)
$$(-1)^{n-1} Lu = \lambda P(t) u$$
$$Tu = 0,$$

where $Tu = 0$ denotes the boundary conditions:

$$u^{(i)}(a) = 0, \quad i = 0, 1, \ldots, k-1$$
$$u^{(i_j)}(b) = 0, \quad j = 1, 2, \ldots, n-k.$$

If $G(t, s)$ is the Green's function for the scalar boundary value problem,

(2)
$$(-1)^{n-1} Ly = 0$$
$$Ty = 0,$$

where Ly and Ty are as above, but defined appropriately for the scalar case, then under certain sign conditions on the $G(t,s)$ and conditions on $P(t)$, we can show the existence of a smallest positive eigenvalue. And with further conditions on $P(t)$, that its corresponding eigenvector is essentially unique with repect to a 'cone'. We also have comparison results for the eigenvalue problems (1) and (3),

$$(3) \qquad (-1)^{n-1}Lu = \Lambda Q(t)u$$
$$Su = 0,$$

where $Su = 0$ are boundary conditions similar to those above.

Our results are new, even in the scalar case. Our technique will be to use the theory of u_o-positive operators with respect to a cone in a Banach space. We then will use sign conditions on a Green's function and then appropriately define an integral operator which will map a cone back into itself. The theory of operators acting on a cone, can be found in books by Krasnosel'skiǐ [12], Deimling [2], and Guo and Lakshmikantham [5]. Related papers include those of Eloe and Henderson [3], Gentry and Travis [4], Hankerson and Peterson [6,7], Kreǐn and Rutman [13], Keener and Travis [10,11], Kreith [14], Schmitt and Smith [16], Tomastik [17,18], and Travis [19].

II) THE GREEN'S FUNCTION:

In this section we give sufficient conditions for the existence and give an explicit form for the Green's function for our problem (2). Also, we will give

certain sign conditions of the Green's function. Some of these sign conditions are new. Theorems 2.1 and 2.2 and Lemmas 2.3, 2.4 and 2.5 are a result of work by Peterson and Riddenhour [15]. We present their work without proof.

We will need the following definition.

Definition: The differential equation $Ly = 0$ is called *right disfocal* on an interval I if there does not exist a nontrivial solution y of $Ly = 0$ and points $t_1 \le t_2 \le \cdots \le t_n \in I$ such that $y^{(i-1)}(t_i) = 0$ for $i = 1, 2, \ldots, n$.

We will also need to introduce some notation. For each fixed s in the interval $[a, b]$, let $\{y_0(t,s), y_1(t,s), \ldots, y_{n-1}(t,s)\}$ be the set of (linearly independent) solutions of $Ly = 0$, satisfying the intial conditions:

$$y_k^{(j)}(t,s)|_{t=s} = \delta_{jk}, \quad 0 \le j, k \le n-1,$$

where δ_{jk} is the Kronecker-delta function

$$\delta_{jk} = \begin{cases} 0, & \text{for } j \ne k \\ 1, & \text{for } j = k. \end{cases}$$

THEOREM 2.1. *Let $Ly = 0$ be right disfocal on $[a, b]$. Then, for each fixed $s \in [a, b]$, the Green's function for (2) exists and is given by, for $a \le t \le s$,*

$$G(t,s) = \frac{(-1)^{n-k}}{D} \begin{vmatrix} 0 & y_k(t,a) & \cdots & y_{n-1}(t,a) \\ y_{n-1}^{(i_1)}(b,s) & y_k^{(i_1)}(b,a) & \cdots & y_{n-1}^{(i_1)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(i_{n-k})}(b,s) & y_k^{(i_{n-k})}(b,a) & \cdots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}.$$

If $s \le t \le b$, then we replace the zero in the first row, first column by $y_{n-1}(t,s)$ with everything else remaining the same.

In the above formula, D is given by:

$$D = \begin{vmatrix} y_k^{(i_1)}(b,a) & y_{k+1}^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ y_k^{(i_2)}(b,a) & y_{k+1}^{(i_2)}(b,a) & \dots & y_{n-1}^{(i_2)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_k^{(i_{n-k})}(b,a) & y_{k+1}^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}.$$

To present the next theorem, we need to consider the following partition of n-tuples. We will say that $(i_1, i_2, \dots, i_n) < (j_1, j_2, \dots, j_n)$ if there exists an integer m such that

i) $i_k = j_k$ for $k = 1, 2, \dots, m-1$

ii) $i_m < j_m$

iii) $i_k \leq j_k$ for $k = m+1, m+2, \dots, n$.

We can now give a comparison theorem and sign conditions on the Green's functions from different boundary value problems.

THEOREM 2.2. *Let $Ly = 0$ be right disfocal on $[a, b]$, and suppose that $(i_1, i_2, \dots, i_{n-k}) < (j_1, j_2, \dots, j_{n-k})$ where $0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1$. If $G_{i_1 \cdots i_{n-k}}(t,s)$ is the Green's function for*

$$Ly = 0$$

$$y^{(i)}(a) = 0, \quad i = 0, 1 \dots, k-1$$

$$y^{(i_j)}(b) = 0, \quad j = 1, 2, \dots, n-k,$$

and $G_{j_1\cdots j_{n-k}}(t,s)$ *is the Green's function for*

$$Ly = 0$$

$$y^{(i)}(a) = 0, \quad i = 0, 1, \ldots, k-1$$

$$y^{(j_i)}(b) = 0, \quad i = 1, 2, \ldots, n-k,$$

then $G^{(p)}_{i_1\cdots i_{n-k}}(t,s) < G^{(p)}_{j_1\cdots j_{n-k}}(t,s)$ *on* $(a,b)^2$ *for* $p = 0, 1, \ldots, i_1$.

We note that the above theorem gives us a sign condition on $G(t,s)$. Since it is well known that $G_{01\cdots n-k-1}(t,s) > 0$ on $(a,b)^2$ we have that if $(i_1, i_2, \ldots, i_{n-k}) > (0, 1, \ldots, n-k)$, then $G_{i_1\cdots i_{n-k}}(t,s) > 0$ on $(a,b)^2$.

The above two theorems are proved using the following lemmas.

LEMMA 2.3. *Let* L^* *be the adjoint operator defined by* $L^*z = z^{(n)} + (-1)^n r(t)z$, *corresponding to our operator* $Ly = y^{(n)} + r(t)y$. *Then* $Ly = 0$ *is right disfocal if and only if* $L^*z = 0$ *is right disfocal.*

Our next lemma gives a relation between our set of solutions to $Ly = 0$ and a set of solutions to $L^*z = 0$.

LEMMA 2.4. *For* $i = 0, 1, \ldots, n-1$, *let* $z_i(t,s)$ *be solutions on* $[a,b]$, *to*

$$L^*z = 0$$

$$z_i^{(j)}(t,s)|_{t=s} = \delta_{ij}, \quad 0 \le j \le n-1.$$

Then, for $0 \le i, j \le n-1$, *we have*

$$y_i^{(j)}(t,s) = (-1)^{i+j} z_{n-j-1}^{(n-i-1)}(s,t)$$

for all $t, s \in [a, b]$.

This lemma is proved by applying the Lagrange identity to $y_i(t, \tau)$ and $z_{n-j-1}(t, s)$ and evaluating at $t = s$ and $t = \tau$.

By using the adjoint realtions of Lemma 2.4, it is easy to show that for any fixed t,

$$(4) \qquad (\frac{\partial}{\partial s})^{\tau}\{y_j^{(i)}(t,s)\} = (-1)^{\tau} y_{j-\tau}^{(i)}(t,s)$$

on $[a, b] \times [a, b]$ for $0 \leq i,\ j \leq n-1,\ 0 \leq \tau \leq j$.

The next lemma is a crux to all of our results.

LEMMA 2.5. *If* $Ly = 0$ *is right disfocal, and* y *is a nontrivial solution to (2), then* $y(t) \neq 0$ *for all* $t \in (a, b)$.

This last lemma is proved by assuming that there exists a $t_\circ \in (a, b)$ and a nontrivial solution y, to $Ly = 0$, such that $y(t_\circ) = 0$. Then using the boundary conditions with a Rolle's Theorem argument, one can contradict $Ly = 0$ is right disfocal. It is important to note that this lemma also holds for the adjoint equation.

We can now give a sign condition on certain derivatives of the Green's function at the end points. To establish the sign condition at $t = b$, we need a bit more notation. Consider our boundary conditions at $t = b$. Suppose that $i_1 > 0$, then we define $k_\circ = 0$. If $i_1 = 0$, then we define $k_\circ$, $1 \leq k_\circ \leq n - k$ to be such that

$i_j = j-1$, for $j = 1, 2, \ldots, k_\circ - 1$ and $k_\circ < i_{k_\circ+1}$ (if $k_\circ < n-k$). So, for example, if we have the $(n-k)$-tuple $(i_1, i_2, \ldots, i_{n-k}) = (0, 1, 2, 5, \ldots, i_{n-k})$, then $k_\circ = 3$.

THEOREM 2.6. *Let $Ly = 0$ be right disfocal and $G(t,s)$ be the Green's function for (2). Then $G^{(k)}(a,s) > 0$ for all $s \in (a,b)$. Further, we define $k_\circ$ as above, then $(-1)^{k_\circ} G^{(k_\circ)}(b,s) > 0$ for all $s \in (a,b)$.*

PROOF: We will first show that $G^{(k)}(a,s) > 0$, for all $s \in (a,b)$. After taking k derivatives and evaluating at $t = a$, we have that the first row $\vec{\mathbf{R}}_1$, of $G^{(k)}(a,s)$ is $\vec{\mathbf{R}}_1 = (0, y_k^{(k)}(a,a), y_{k+1}^{(k)}(a,a), \ldots, y_{n-1}^{(k)}(a,a)) = (0, 1, 0, \ldots, 0)$. Now, define $f(s)$ on an open interval which contains $[a,b]$ by

$$f(s) = \begin{vmatrix} y_{n-1}^{(i_1)}(b,s) & y_{k+1}^{(i_1)}(b,a) & \ldots & y_{n-1}^{(i_1)}(b,a) \\ y_{n-1}^{(i_2)}(b,s) & y_{k+1}^{(i_2)}(b,a) & \ldots & y_{n-1}^{(i_2)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(i_{n-k})}(b,s) & y_{k+1}^{(i_{n-k})}(b,a) & \ldots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}.$$

So we have that $G^{(k)}(a,s) = \frac{(-1)^{n-k+1}}{D} f(s)$ on $[a,b]$. Now, we can show that $f(s) \neq 0$ for all $s \in (a,b)$. We first transform f into its equivalent adjoint form using Lemma 2.4. Then f satisfies $L^*z = 0$ and the equivalent adjoint boundary conditions. So from the adjoint form of Lemma 2.5, we have that $f(s) \neq 0$ for all $s \in (a,b)$.

Now, consider any element in the first column of f. By using (4), we have that $(\frac{\partial}{\partial s})^j \{y_{n-1}^{(i)}(b,s)\}|_{s=b} = (-1)^j y_{n-1-j}^{(i)}(b,b) = 0$, for $0 \leq j < (n-1) - i_{n-k}$ and $i \in \{i_1, i_2, \ldots, i_{n-k}\}$. (If $i_{n-k} = n-1$ then we define $j = 0$.) This tells us that $f^{(j)}(b) = 0$ for $0 \leq j < (n-1) - i_{n-k}$. Now, letting $j = n - 1 - i_{n-k}$, we

have that

$$f^{(j)}(s) = (-1)^j \begin{vmatrix} y_{i_{n-k}}^{(i_1)}(b,b) & y_{k+1}^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ y_{i_{n-k}}^{(i_2)}(b,b) & y_{k+1}^{(i_2)}(b,a) & \dots & y_{n-1}^{(i_2)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_{i_{n-k}}^{(i_{n-k})}(b,b) & y_{k+1}^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}$$

$$= (-1)^j \begin{vmatrix} 0 & y_{k+1}^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ 0 & y_{k+1}^{(i_2)}(b,a) & \dots & y_{n-1}^{(i_2)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_{k+1}^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}$$

$$= (-1)^j(-1)^{n-k+1} \begin{vmatrix} y_{k+1}^{(i_1)}(b,a) & y_{k+2}^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ y_{k+1}^{(i_2)}(b,a) & y_{k+2}^{(i_2)}(b,a) & \dots & y_{n-1}^{(i_2)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_{k+1}^{(i_{n-k})}(b,a) & y_{k+2}^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}.$$

It is a standard argument to show that the above determinant is positive. This gives us that $(-1)^j(-1)^{n-k+1}f^{(j)}(b) > 0$. Now, since $f^{(i)}(b) = 0$, for $0 \le i \le j$, we can use a Taylor series expansion on $(-1)^{n-k+1}f(s)$, about b, to get that

$$(-1)^{n-k+1}f(s) = (-1)^{n-k+1}f^{(j)}(b)\frac{(x-b)^j}{j!} + O((x-b)^{j+1}).$$

This tells us that for a sufficiently small $\delta > 0$, if j is even, then $(-1)^{n-k+1}f(s) > 0$ for all $s \in (b-\delta, b)$. If j is odd, then $(-1)^{n-k+1}f(s) < 0$ for all $s \in (b-\delta, b)$, or $(-1)^j\{(-1)^{n-k+1}f(s)\} > 0$ for all $s \in (b-\delta, b)$. In either case, for a small enough δ, we have that $(-1)^{n-k+1}f(s) > 0$ for all $s \in (b-\delta, b)$. But, we have already shown that f is of one sign on (a,b). Thus $(-1)^{n-k+1}f(s) > 0$ for all $s \in (a,b)$. This gives us that $G^{(k)}(a,s) = \frac{(-1)^{n-k+1}}{D}f(s) > 0$ for all $s \in (a,b)$, and so the first part of our theorem is proved.

To prove our sign condition at $t = b$, we suppose that $i_1 = 0$ and $k_\circ$ is defined as before. We define the function f on an open interval containing $[a, b]$ by

$$f(s) = \frac{(-1)^{n-k}}{D} \begin{vmatrix} y_{n-1}^{(k_\circ)}(b,s) & y_k^{(k_\circ)}(b,a) & \dots & y_{n-1}^{(k_\circ)}(b,a) \\ y_{n-1}^{(i_1)}(b,s) & y_k^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(i_{n-k})}(b,s) & y_k^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}.$$

By defining f in this manner, we have that $G^{(k_\circ)}(b,s) = \frac{(-1)^{n-k}}{D} f(s)$ for $s \in [a, b]$. Like before, we show that $f(s) \neq 0$ for all $s \in (a, b)$ by first transforming f into its equivalent adjoint form using Lemma 4. Then f satisfies $L^* z = 0$ and the equivalent adjoint boundary conditions. So from the adjoint form of Lemma 5, we must have that $f(s) \neq 0$ for all $s \in (a, b)$.

Consider any element in the first column of f. By using (4), we have that

$$(\frac{\partial}{\partial s})^j \{y_{n-1}^{(i)}(b,s)\}|_{s=b} = (-1)^j y_{n-1-j}^{(i)}(b,b) = 0,$$

for $0 \leq j < (n-1) - i_{n-k}$ and $i \in \{i_1, i_2, \dots, i_{n-k}\}$. (If $i_{n-k} = n-1$ then we define $j = 0$.) This tells us that $f^{(j)}(b) = 0$ for $0 \leq j < (n-1) - i_{n-k}$. Now,

letting $j = (n-1) - i_{n-k}$, we have that

$$f^{(j)}(b) = (-1)^j \begin{vmatrix} y_{i_{n-k}}^{(k_\circ)}(b,b) & y_k^{(k_\circ)}(b,a) & \dots & y_{n-1}^{(k_\circ)}(b,a) \\ y_{i_{n-k}}^{(i_1)}(b,b) & y_k^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_{i_{n-k}}^{(i_{n-k})}(b,b) & y_k^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}$$

$$= (-1)^j \begin{vmatrix} 0 & y_k^{(k_\circ)}(b,a) & \dots & y_{n-1}^{(k_\circ)}(b,a) \\ 0 & y_k^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_k^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}$$

$$= (-1)^j(-1)^{n-k} \begin{vmatrix} y_k^{(k_\circ)}(b,a) & y_{k+1}^{(k_\circ)} & \dots & y_{n-1}^{(k_\circ)}(b,a) \\ y_k^{(i_1)}(b,a) & y_{k+1}^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_k^{(i_{n-k})}(b,a) & y_{k+1}^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}$$

$$= (-1)^j(-1)^{n-k}(-1)^{k_\circ} \begin{vmatrix} y_k^{(i_1)}(b,a) & y_{k+1}^{(i_1)}(b,a) & \dots & y_{n-1}^{(i_1)}(b,a) \\ y_k^{(i_2)}(b,a) & y_{k+1}^{(i_2)}(b,a) & \dots & y_{n-1}^{(i_2)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_k^{(i_{k_\circ})}(b,a) & y_{k+1}^{(i_{k_\circ})}(b,a) & \dots & y_{n-1}^{(i_{k_\circ})}(b,a) \\ y_k^{(k_\circ)}(b,a) & y_{k+1}^{(k_\circ)}(,a) & \dots & y_{n-1}^{(k_\circ)}(b,a) \\ y_k^{(i_{k_\circ}+1)}(b,a) & y_{k+1}^{(i_{k_\circ}+1)}(b,a) & \dots & y_{n-1}^{(i_{k_\circ}+1)}(b,a) \\ \vdots & \vdots & \ddots & \vdots \\ y_k^{(i_{n-k})}(b,a) & y_{k+1}^{(i_{n-k})}(b,a) & \dots & y_{n-1}^{(i_{n-k})}(b,a) \end{vmatrix}.$$

Now, from our construction of $k_\circ$, we have that $i_{k_\circ} < k_\circ < i_{k_\circ}$, so again from a standard argument we have that the above determinant is strictly greater that zero. This gives us that $(-1)^j(-1)^{n-k}(-1)^{k_\circ} f^{(j)}(b) > 0$. Now, since $f^{(i)}(b) = 0$, for $0 \le i \le j$, we can again use a Taylor series expansion on $(-1)^{n-k}(-1)^{k_\circ} f(s)$,

about b, to get

$$(-1)^{n-k}(-1)^{k_\circ}f(s) = (-1)^{n-k}(-1)^{k_\circ}f^{(j)}(b)\frac{(x-b)^j}{j!} + O((x-b^{j+1})).$$

From this we can again see that for $\delta > 0$ sufficiently small, that if j is even, then $(-1)^{n-k}(-1)^{k_\circ}f(s) > 0$. If j is odd, then $(-1)^{n-k}(-1)^{k_\circ}f(s) < 0$ for all $s \in (b-\delta, b)$, and so $(-1)^j\{(-1)^{n-k}(-1)^{k_\circ}f(s)\} > 0$. In either case we have that $(-1)^{n-k}(-1)^{k_\circ}f(s) > 0$ for all $s \in (b-\delta, b)$ for small enough δ. But, we have already shown that f is of one sign on (a,b). Thus $(-1)^{n-k}(-1)^{k_\circ}f(s) > 0$ for all $s \in (a,b)$. This gives us that $(-1)^{k_\circ}G^{(k_\circ)}(b,s) = \frac{(-1)^{n-k}}{D}(-1)^{k_\circ}f(s) > 0$ for all $s \in (a,b)$, and so our theorem is proved.

IV) EXISTENCE AND COMPARISON RESULTS:

We will now introduce a suitable Banach space for our eigenvalue problem (1). Recall that the boundary conditions $Tu = 0$, for u an m-column vector, are $u^{(i)}(a) = 0$, for $i = 0, 1, \ldots, k-1$, and $u^{(i_j)}(b) = 0$, for $j = 1, 2, \ldots, n-k$, where $0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1$. First, let us suppose that $i_1 \neq 0$. When $i_1 \neq 0$, we will denote these boundary conditions as $T_1 u = 0$.

We now introduce the Banach space

$$\mathcal{B}_1 = \{u \in C^n([a,b], \mathcal{R}^m) \,|\, u^{(i)}(a) = 0, i = 0, 1, \ldots, k-1\}$$

with norm $||u|| = \max_{0 \leq i \leq n}\{\max_{[a,b]} |u^{(i)}(t)|\}$ where $|\cdot|$ is the Euclidean norm. Following ideas from Hankerson and Peterson [5, 6], and Tomastik's paper [13],

we let $I, J \subseteq \{1, 2, \ldots, m\}$ be such that $I \cup J = \{1, 2, \ldots, m\}$ and $I \cap J = \emptyset$. (It is permissible for $I = \emptyset$ or $J = \emptyset$.) Let $\mathcal{K}$ be the 'quadrant' cone in $\mathcal{R}^m$ defined by

$$\mathcal{K} = \{x = (x_1, \ldots, x_m) \,|\, x_i \geq 0 \text{ if } i \in I,\; x_i \leq 0 \text{ if } i \in J\}.$$

Although some of our results will hold for any solid cone in $\mathcal{R}^m$, we will just concern ourselves with $\mathcal{K}$ being a 'quadrant' cone in $\mathcal{R}^m$. Define δ_i to be the discrete function $\delta_i = 1$ if $i \in I$, and $\delta_i = -1$ if $i \in J$. We can then equivalently define the cone $\mathcal{K}$ to be $\mathcal{K} = \{x \in \mathcal{R}^m \,|\, \delta_i x_i \geq 0 \text{ for } i = 1, 2, \ldots, m\}$. This also allows us to define the interior of $\mathcal{K}$ as $\mathcal{K}^\circ = \{x \in \mathcal{R}^m \,|\, \delta_i x_i > 0 \text{ for } i = 1, 2, \ldots, m\}$.

We now define the reproducing cone $\mathcal{P}_1 \subset \mathcal{B}_1$ by $\mathcal{P}_1 = \{u \in \mathcal{B}_1 \,|\, u(t) \in \mathcal{K},\, t \in [a, b]\}$. This gives us the following Lemma concerning the interior of our cone $\mathcal{P}_1$.

LEMMA 2.7. *Let the cone $\mathcal{P}_1$ in the Banach space $\mathcal{B}_1$ be defined as above. Then the interior of $\mathcal{P}_1$ is given by*

$$\mathcal{P}_1^\circ = \{u \in \mathcal{B}_1 \,|\, u(t) \in \mathcal{K}^\circ,\, t \in (a, b] \text{ and } u^{(k)}(a) \in \mathcal{K}^\circ\},$$

or equivalently

$$\mathcal{P}_1^\circ = \{u \in \mathcal{B}_1 \,|\, \delta_i u_i(t) > 0,\, t \in (a, b] \text{ and } \delta_i u_i^{(k)}(a) > 0,\, i = 1, 2, \ldots, m\}.$$

PROOF: Let $Q = \{u \in \mathcal{B}_1 \,|\, u(t) \in \mathcal{K}^\circ,\, t \in (a, b] \text{ and } u^{(k)}(a) \in \mathcal{K}^\circ\}$. First we will show that $Q \subseteq \mathcal{P}_1^\circ$. Let u be an arbitrary element of Q, so we want to find an $\varepsilon > 0$

so that the ball $B(u;\varepsilon) \subset \mathcal{P}_1$. For a vector function $x(t)$ on $[\alpha,\beta] \subseteq [a,b]$ we define the distance function $d_{[\alpha,\beta]}(x(t),\partial\mathcal{K})$ to be the distance between the function $x(t)$ on $[\alpha,\beta]$ and the boundary of the cone $\partial\mathcal{K}$. Let $\varepsilon_1 = \frac{1}{2}d_{[a,b]}(u^{(k)}(a),\partial\mathcal{K})$, so we have that $\varepsilon_1 > 0$ since $u^{(k)}(a) \in \mathcal{K}^\circ$. Now $u^{(k)}$ is a continuous function, so there exists a $\delta > 0$ so that $u^{(k)}(t) \in B(u^{(k)}(a);\varepsilon_1) \subset \mathcal{R}^m$, for all $t \in [a,a+\delta]$. We note that this gives us that $d_{[a,a+\delta]}(u^{(k)}(t),\partial\mathcal{K}) > \varepsilon_1$.

Thus we have that $u(t) \in \mathcal{K}^\circ$ for all $t \in [a+\delta,b]$. Then, if we now let $\varepsilon_2 = \frac{1}{2}d_{[a+\delta,b]}(u(t),\partial\mathcal{K})$ we also have that $\varepsilon_2 > 0$ since the graph of $u(t)$, which is compact on $[a+\delta,b]$, and $\partial\mathcal{K}$ do not intersect. We note that in this case, we have that $d_{[a+\delta,b]}(u(t),\partial\mathcal{K}) > \varepsilon_2$.

Let $\varepsilon = min\{\varepsilon_1,\varepsilon_2\} > 0$. Then we have that $B(u;\varepsilon) \subset \mathcal{P}_1$. To show this, we let $z \in B(u;\varepsilon)$. Then $||z-u|| < \varepsilon$ so in particular we have that $|z^{(k)}(a)-u^{(k)}(a)| < \varepsilon_1 = \frac{1}{2}d_{[a,b]}(u^{(k)}(a),\partial\mathcal{K})$. This tells us that $z^{(k)}(a) \in \mathcal{K}^\circ$. Now $||z-u|| < \varepsilon$ also tells us that $|z^{(k)}(t) - u^{(k)}(t)| < \varepsilon$ for all $t \in [a,a+\delta]$. This gives us that $z^{(k)}(t) \in \mathcal{K}^\circ$ for all $t \in [a,a+\delta]$. If this were not so, then since $z^{(k)}(a) \in \mathcal{K}^\circ$ and $z^{(k)}$ is continuous, there would exists a $t_\circ \in [a,a+\delta]$ so that $z^{(k)}(t_\circ) \in \partial\mathcal{K}$. But from the note above we know that $d_{[a,a+\delta]}(u^{(k)}(t),\partial\mathcal{K}) > \varepsilon_1 \geq \varepsilon$. This gives us that $|z^{(k)}(t_\circ) - u^{(k)}(t_\circ)| \geq \varepsilon$ which is a contradiction. Thus $z^{(k)}(t) \in \mathcal{K}^\circ$ for all $t \in [a,a+\delta]$.

Now, this last statement tells us that for $i = 1,2,\ldots,m$, $\delta_i z_i^{(k)}(t) > 0$ for all $t \in [a,a+\delta]$. Thus $\delta_i z_i^{(k-1)}(t)$ is a strictly increasing function on $[a,a+\delta]$ with

$\delta_i z_i^{(k-1)}(a) = 0$ for each i. Hence we have that $\delta_i z_i^{(k-1)}(t) > 0$ for all $t \in (a, a+\delta]$, for $i = 1, 2, \ldots, m$. Thus $\delta_i z_i^{(k-2)}(t)$ is strictly increasing on $(a, a+\delta]$, with $\delta_i z_i^{(k-2)}(a) = 0$ for each i. Thus $\delta_i z_i^{(k-2)}(t) > 0$ on $(a, a+\delta]$ for each i. Hence, for each $i = 1, 2, \ldots, m$, we have that $\delta_i z_i^{(k-3)}(t)$ is strictly increasing on $(a, a+\delta]$ with $\delta_1 z_i^{(k-3)}(a) = 0$. Continuing in this manner, we eventually come to the conclusion that $z(t) \in \mathcal{K}$ for $t \in [a, a+\delta]$.

Also, we have that $|z(t)-u(t)| < \varepsilon \leq \varepsilon_2$ for all $t \in [a+\delta, b]$. Thus, $z(t) \notin \partial\mathcal{K}$ or else we contradict $\varepsilon_2 < d_{[a+\delta,b]}(u(t), \partial\mathcal{K})$. Since $z(a+\delta) \in \mathcal{K}^\circ$ and z is continuous, we must have that $z(t) \in \mathcal{K}^\circ$ for all $t \in [a+\delta, b]$

Thus $z(t) \in \mathcal{K}$ for all $t \in [a, b]$. But this means that $z \in \mathcal{P}_1$, and since z was an arbitrary element of $B(u;\varepsilon)$, we have that $B(u;\varepsilon) \subset \mathcal{P}_1$. But u was an arbitrary element of Q and we found an $\varepsilon > 0$ so that $B(u;\varepsilon) \subset \mathcal{P}_1$. Thus we have that $Q \subseteq \mathcal{P}_1^\circ$.

We now show that $\mathcal{P}_1^\circ \subseteq Q$. Let u be an arbitrary element of $\mathcal{P}_1^\circ$. Suppose there exists a $t_\circ \in (a, b]$ so that $u(t_\circ) \in \partial\mathcal{K}$. This give us that there exists a component of u, say $u_{i_\circ}$, so that $u_{i_\circ}(t_\circ) = 0$. Considering the scalar equation, $\delta_{i_\circ} u_{i_\circ}(t) > 0$, it can be seen that for any $\varepsilon > 0$, since $\delta_{i_\circ} u_{i_\circ}(t_\circ) = 0$, we can find a function $\delta_{i_\circ} z_{i_\circ}(t) \in B(\delta_{i_\circ} u_{i_\circ}; \varepsilon)$ so that $\delta_{i_\circ} z_{i_\circ}(t_\circ) < 0$. If we let the vector function $z(t)$ equal $u(t)$ in each component except in the $i_\circ$ slot, and then in that slot let $(z(t))_{i_\circ} = z_{i_\circ}(t)$, then $z \in B(u;\varepsilon)$. But $z(t_\circ) \notin \mathcal{K}$ since $\delta_{i_\circ} z_{i_\circ}(t_\circ) < 0$. Thus $z \notin \mathcal{P}_1$. Now z was based on $\varepsilon > 0$. Thus, for any $\varepsilon > 0$ we can find a

$z \in B(u;\varepsilon)$ and $z \notin \mathcal{P}_1$. This contradicts $u \in \mathcal{P}_1^\circ$. Thus $u(t) \in \mathcal{K}^\circ$ for all $t \in (a,b]$.

Now suppose $u^{(k)}(a) \notin \mathcal{K}^\circ$. So there exists an i so that $\delta_i u_i^{(k)}(a) \leq 0$. Then for any $\varepsilon > 0$ we can find a $z \in B(u;\varepsilon)$ so that $\delta_i z_i^{(k)}(a) < 0$. Thus $\delta_i z_i^{(k)}$ is strictly decreasing at a. We have that $z_i^{(k-1)}(a) = 0$ so we can find a $\delta > 0$ so that $\delta_i z_i^{(k-1)}(t) < 0$ for any $t \in (a, a+\delta]$. Thus, $\delta_i z^{(k-2)}$ is strictly decreasing on $(a, a+\delta]$ and $\delta_i z^{(k-2)}(a) = 0$. Hence $\delta_i z^{(k-2)}(t) < 0$ for all $t \in (a, a+\delta]$. Like before, by continuing in this manner we come to the conclusion that $z(t_\circ) \notin \mathcal{K}$ and so $z \notin \mathcal{P}_1$, which contradicts $u \in \mathcal{P}_1^\circ$. Thus we must have that $u^{(k)}(a) \in \mathcal{K}^\circ$.

So if $u \in \mathcal{P}_1^\circ$ we have that $u(t) \in \mathcal{K}^\circ$ for all $t \in (a,b]$, and also that $u^{(k)}(a) \in \mathcal{K}^\circ$. Thus $u \in Q$, and since u was an arbitrary element of $\mathcal{P}_1^\circ$, we have that $\mathcal{P}_1^\circ \subseteq Q$. Thus our lemma is proved.

Now let us suppose that $i_1 = 0$. We will denote these boundary conditions as $T_0 u = 0$. As in the last section, let $k_\circ$, $1 \leq k_\circ \leq n-k$, be such that $i_j = j-1$ for $j = 1, 2, k_\circ - 1$ and $i_{k_\circ - 1} < k_\circ < i_{k_\circ}$ (if $k_\circ < n-k$).

We now introduce the Banach space

$$\mathcal{B}_0 = \{u \in C^n([a,b], \mathcal{R}^m) \,|\, u^{(i)}(a) = 0, \, , 0 \leq i \leq k-1, \; u^{(i)}(b) = 0, \; 0 \leq i \leq k_\circ - 1\},$$

with norm $||u|| = \max_{0 \leq i \leq n}\{\max_{[a,b]} |u^{(i)}(t)|\}$ where $|\cdot|$ is the Euclidean norm.

We now define the reproducing cone $\mathcal{P}_0 \subset \mathcal{B}_0$ by $\mathcal{P}_0 = \{u \in \mathcal{B}_0 \,|\, u(t) \in \mathcal{K}, \; t \in [a,b]\}$. We also have a lemma concerning the interior of this cone $\mathcal{P}_0$.

LEMMA 2.8. *Let the cone $\mathcal{P}_0$ in the Banach space $\mathcal{B}_0$ be defined as above. Then*

the interior of $\mathcal{P}_0$ is given by

$$\mathcal{P}_0^\circ = \{u \in \mathcal{B}_0 \,|\, u(t) \in \mathcal{K}^\circ,\, t \in (a,b),\, u^{(k)}(a) \in \mathcal{K}^\circ, \text{ and } (-1)^{k_\circ} u^{(k_\circ)}(b) \in \mathcal{K}^\circ\}$$

or equivalently

$$\mathcal{P}_0^\circ = \{u \in \mathcal{B}_0 \,|\, \delta_i u_i(t) > 0,\, t \in (a,b),\, \delta_i u_i^{(k)}(a) > 0, \text{ and}$$
$$(-1)^{k_\circ} \delta_i u_i^{(k_\circ)}(b) > 0,\, 1 \leq i \leq m\}.$$

PROOF: The proof of this lemma is very similiar to the proof of Lemma 2.7. Let $Q = \{u \in \mathcal{B}_0 \,|\, \delta_i u_i(t) > 0,\, t \in (a,b),\, \delta_i u_i^{(k)}(a) > 0,\, (-1)^{k_\circ}\delta_i u_i^{(k_\circ)}(b) > 0,\, 1 \leq i \leq m\}$. First we will show that $Q \subseteq \mathcal{P}_0^\circ$. Let u be an arbitrary element of Q, so we want to find an $\varepsilon > 0$ so that the ball $B(u;\varepsilon) \subset \mathcal{P}_1$. Now, from the argument in Lemma 2.7, we see that if we let $\varepsilon_1 = \frac{1}{2} d_{[a,b]}(u^{(k)}(a), \partial\mathcal{K}) > 0$, then there exists a $\delta_1 > 0$ such that $u^{(k)}(t) \in B(u^{(k)}(a);\varepsilon_1) \subset \mathcal{R}^m$, for all $t \in [a, a+\delta_1]$.

If we let $\varepsilon_2 = \frac{1}{2} d_{[a,b]}((-1)^{k_\circ} u^{(k_\circ)}(b), \partial\mathcal{K}) > 0$, then there exists a $\delta_2 > 0$ such that $(-1)^{k_\circ} u^{(k_\circ)}(t) \in B((-1)^{k_\circ} u^{(k)}(a);\varepsilon_2) \subset \mathcal{R}^m$, for all $t \in [b-\delta_2, b]$. We note that this gives us that $d_{[b-\delta_2,b]}((-1)^{k_\circ} u^{(k)}(t), \partial\mathcal{K}) > \varepsilon_2$.

Since $u(t) \in \mathcal{K}^\circ$ for all $t \in [a+\delta_1, b-\delta_2]$ if we let $\varepsilon_3 = \frac{1}{2} d_{[a+\delta_1, b-\delta_2]}(u(t), \partial\mathcal{K})$, then we have that $\varepsilon_3 > 0$ since the compact graph of $u(t)$ on $[a+\delta_1, b-\delta_2]$, and $\partial\mathcal{K}$ do not intersect.

Let $\varepsilon = min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} > 0$. Then we have that $B(u;\varepsilon) \subset \mathcal{P}_0$. To show this, we let $z \in B(u;\varepsilon)$. Then, similiar to the arguments in Lemma 2.7, we have that $z(t) \in \mathcal{K}$ for all $t \in [a, b-\delta]$, for some $\delta > 0$.

Also, since $||z - u|| < \varepsilon$, we have in particular we have that $|z^{(k_\circ)}(b) - u^{(k_\circ)}(b)| = |(-1)^{k_\circ} z^{(k_\circ)}(b) - (-1)^{k_\circ} u^{(k_\circ)}(b)| < \varepsilon_2$. Then since $(-1)^{k_\circ} u^{(k_\circ)}(b) \in \mathcal{K}^\circ$ and

$\varepsilon_2 = \frac{1}{2} d_{[a,b]}((-1)^{k_\circ} u^{(k_\circ)}(b), \partial\mathcal{K}) > 0$, we have that $(-1)^{k_\circ} z^{(k_\circ)}(b) \in \mathcal{K}^\circ$.

Now $||z - u|| < \varepsilon$ also tells us that $|(-1)^{k_\circ} z^{(k_\circ)}(t) - (-1)^{k_\circ} u^{(k_\circ)}(t)| < \varepsilon$ for all $t \in [b - \delta_2, b]$. This gives us that $(-1)^{k_\circ} z^{(k_\circ)}(t) \in \mathcal{K}^\circ$ for all $t \in [b - \delta_2, b]$. If this were not so, then since $(-1)^{k_\circ} z^{(k_\circ)}(b) \in \mathcal{K}^\circ$ and $(-1)^{k_\circ} z^{(k_\circ)}$ is continuous, there would exists a $t_\circ \in [b - \delta_2, b]$ so that $(-1)^{k_\circ} z^{(k_\circ)}(t_\circ) \in \partial\mathcal{K}$. But from the note above we know that $d_{[b-\delta_2,b]}((-1)^{k_\circ} u^{(k)}(t), \partial\mathcal{K}) > \varepsilon_2 \geq \varepsilon$. This gives us that $|z^{(k_\circ)}(t_\circ) - u^{(k_\circ)}(t_\circ)| \geq \varepsilon$ which is a contradiction. Thus $(-1)^{k_\circ} z^{(k_\circ)}(t) \in \mathcal{K}^\circ$ for all $t \in [b - \delta, b]$.

This last statement tells us that for $i = 1, 2, \ldots, m$, $\delta_i (-1)^{k_\circ} z_i^{(k_\circ)}(t) > 0$ for all $t \in [b - \delta, b]$. Now, since $z_i^{(j)}(b) = 0$ for $j = 0, 1, \ldots, k_\circ - 1$, $i = 1, 2, \ldots, m$, we can use a Taylor series argument (as in the proof of Theorem 6), to show that $\delta_i z_i(t) \geq 0$ for all $t \in [b - \delta_2, b]$, $i = 1, 2, \ldots, m$. Hence $z(t) \in \mathcal{K}$ for all $t \in [b - \delta_2, b]$.

Combining our cases we have that $z(t) \in \mathcal{K}$ for all $t \in [a, b]$. But this means that $z \in \mathcal{P}_0$, and since z was an arbitrary element of $B(u; \varepsilon)$, we have that $B(u; \varepsilon) \subset \mathcal{P}_0$. But u was an arbitrary element of Q and we found an $\varepsilon > 0$ so that $B(u; \varepsilon) \subset \mathcal{P}_1$. Thus we have that $Q \subseteq \mathcal{P}_0^\circ$.

We now show that $\mathcal{P}_0^\circ \subseteq Q$. Let u be an arbitrary element of $\mathcal{P}_0^\circ$. Now, following arguments similiar to the ones in Lemma 7, we see that $u(t) \in \mathcal{K}^\circ$ for

all $t \in (a, b)$, and that $u^{(k)}(a) \in \mathcal{K}^\circ$.

Now suppose $(-1)^{k_\circ} u^{(k_\circ)}(b) \notin \mathcal{K}^\circ$. So there exists an i so that $\delta_i(-1)^{k_\circ} u^{(k_\circ)}(b) \leq 0$. Then for any $\varepsilon > 0$ we can find a $z \in B(u; \varepsilon)$ so that $\delta_i(-1)^{k_\circ} z_i^{(k_\circ)}(b) < 0$. Now, $z_i^{(j)}(b) = 0$ for $j = 0, 1, \ldots, k_\circ$. So, again using a Taylor series argument, we can show that $\delta_i z_i(t) < 0$ on $(b - \delta, b)$ for a sufficiently small $\delta > 0$. But then $z(t) \notin \mathcal{K}$ for $t \in (b - \delta, b)$ which tells us that $z \notin \mathcal{P}_0$. This contradicts $u \in \mathcal{P}_0^\circ$. Hence we must have that $(-1)^{k_\circ} u^{(k_\circ)}(b) \in \mathcal{K}^\circ$.

So if $u \in \mathcal{P}_0^\circ$ we have that $u(t) \in \mathcal{K}^\circ$ for all $t \in (a, b)$, and also that $u^{(k)}(a) \in \mathcal{K}^\circ$ and $(-1)^{k_\circ} u^{(k_\circ)}(b) \in \mathcal{K}^\circ$. Thus $u \in Q$, and since u was an arbitrary element of $\mathcal{P}_0^\circ$, we have that $\mathcal{P}_0^\circ \subseteq Q$. Thus our lemma is proved.

With our Banach spaces and cones suitable defined, we can now proceed on to our first existence result.

THEOREM 2.9. *Let $Ly = 0$ be right disfocal, and assume that $\delta_i \delta_j p_{ij}(t) \geq 0$, for $t \in [a, b]$, $1 \leq i, j \leq m$, and that there is a $t_\circ \in [a, b]$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$. Then for eigenvalue problem (1)*

$$(-1)^{n-1} Lu = \lambda P(t) u$$

$$T_1 u = 0, \quad (so\ (i_1 > 0),$$

there exists an eigenvector $z_\circ \in \mathcal{P}_1$ with corresponding positive eigenvalue $\lambda_\circ$ which is a lower bound for the modulus of any other eigenvalue for the corresponding problem.

PROOF: To solve this problem, we will seek the eigenvalues of the linear integral operator $M\colon \mathcal{B}_1 \to \mathcal{B}_1$ defined by

$$Mu(t) = \int_a^b G(t,s)P(s)u(s)\,ds, \quad a \le t \le b,$$

where $G(t,s)$ is the Green's function for (2). Now the eigenvalues of the boundary value problem (1) are reciprocals of the eigenvalues of the operator M. We note that zero is not an eigenvalue of (1) since $Ly = 0$ is assumed to be right disfocal.

Now an argument using the *Arzela-Ascoli* Theorem shows that M is a compact operator. We now show that $M\colon \mathcal{P}_1 \to \mathcal{P}_1$. Let u be an arbitrary element of $\mathcal{P}_1$. If we can show that $\delta_i(Mu(t))_i \ge 0$ for all $t \in [a,b]$, $i = 1,2,\ldots,m$, then $Mu \in \mathcal{P}_1$. Consider the *ith component* of $P(t)u(t)$, $\big(P(t)u(t)\big)_i = \sum_{j=1}^m p_{ij}(t)u_j(t)$. Now $\delta_j\delta_j = 1$, and $\delta_j u_j(t) \ge 0$ so we have that for all $t \in [a,b]$,

$$\delta_i(P(t)u(t))_i = \sum_{j=1}^m \delta_i\delta_j p_{ij}(t)\delta_j u_j(t) \ge 0,$$

since $\delta_i\delta_j p_{ij}(t) \ge 0$ by hypothesis. From the note following Theorem 2.2, we have that $G(t,s) > 0$ on $(a,b) \times (a,b)$. Thus

$$\delta_i(Mu)_i(t) = \int_a^b G(t,s)\sum_{j=1}^m \delta_i\delta_j p_{ij}(s)\delta_j u_j(s)\,ds \ge 0,$$

for $t \in [a,b]$,$1 \le i \le m$ and so $Mu \in \mathcal{P}_1$. Since u was an arbitrary element of $\mathcal{P}_1$, we have that M is a positive operator, that is $M\colon \mathcal{P}_1 \to \mathcal{P}_1$.

In order to apply Theorem 1.6, we must find a nontrivial $u_\circ \in \mathcal{P}_1$, and an $\varepsilon_\circ > 0$ so that $Mu_\circ \ge \varepsilon_\circ u_\circ$. Let $u_\circ(t) = \frac{(t-a)^k}{k!}\delta_{i_\circ}e_{i_\circ}$, where $e_{i_\circ}$ is the

unit vector in $\mathcal{R}^m$ in the $i_\circ$ direction. We note that $u \in \mathcal{B}_1$. Now the jth component of $u_\circ(t)$ is $u_{\circ j}(t) = \frac{(t-a)^k}{k!}\delta_{i_\circ}\delta_{i_\circ j}$, where δ_{ij} is the Kronecker delta function. Thus $\delta_j u_{\circ j}(t) = \{\delta_j \delta_{i_\circ}\frac{(t-a)^k}{k!}\}\delta_{i_\circ j} \geq 0$, so $u_\circ \in \mathcal{P}_1$. We have that $\delta_{i_\circ} u_{\circ i_\circ}(t) = \frac{(t-a)^k}{k!} > 0$, on $(a, b]$ and that $\delta_{i_\circ} u^{(k)}_{\circ i_\circ}(a) = 1 > 0$.

We now consider $Mu_\circ(t)$. Since $M: \mathcal{P}_1 \to \mathcal{P}_1$, we know that $\delta_j(Mu_\circ)_j(t) \geq 0 = \delta_j u_{\circ j}(t)$ for $1 \leq j \leq m$, $j \neq i_\circ$. When $j = i_\circ$ we have that

$$\begin{aligned}
\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) &= \int_a^b G(t,s)\sum_{j=1}^{m}\delta_{i_\circ}\delta_j p_{i_\circ j}(s)\delta_j u_{\circ j}(s)\,ds \\
&= \int_a^b G(t,s)\delta_{i_\circ}\delta_{i_\circ} p_{i_\circ i_\circ}(s)\delta_{i_\circ} u_{\circ i_\circ}(s)\,ds \\
&= \int_a^b G(t,s)p_{i_\circ i_\circ}(s)\frac{(s-a)^k}{k!}\,ds \\
&> 0, \qquad \text{for } t \in (a,b],
\end{aligned}$$

since by Theorems 2.2 and 2.6, $G(t,s) > 0$ for $t \in (a,b]$, $s \in (a,b)$, and $p_{i_\circ i_\circ}(t_\circ) > 0$, $p_{i_\circ i_\circ}$ continuous. So we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) > 0$ for all $t \in (a,b]$.

Now, Theorem 2.6 tells us that $G^{(k)}(a,s) > 0$ for all $s \in (a,b)$, so we can see from above that $\delta_{i_\circ}(Mu_\circ)^{(k)}_{i_\circ}(a) > 0$. Hence, we can find an $\varepsilon_1 > 0$, sufficiently small, so that $\delta_{i_\circ}(Mu_\circ)^{(k)}_{i_\circ}(a) - \varepsilon_1\delta_{i_\circ}u^{(k)}_{\circ i_\circ}(a) > 0$. Now $\delta_{i_\circ}(Mu_\circ)^{(j)}_{i_\circ}(a) - \varepsilon_1\delta_{i_\circ}u^{(j)}_{\circ i_\circ}(a) = 0$, for $j = 0, 1, \ldots, k-1$. Thus we can find a $\delta > 0$ so that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_1\delta_{i_\circ}u_{\circ i_\circ}(t) \geq 0$, for all $t \in [a, a+\delta]$.

Now both $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t)$ and $\delta_{i_\circ}u_{\circ i_\circ}(t)$ are positive on $[a+\delta, b]$ so we can let

$$\varepsilon_2 = \frac{\min_{[a+\delta,b]}(\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t)}{\max_{[a+\delta,b]}(\delta_{i_\circ}u_{\circ i_\circ}(t)} > 0.$$

This gives us that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_2\delta_{i_\circ}u_{\circ i_\circ}(t) \geq 0$, for all $t \in [a+\delta, b]$.

Finally, letting $\varepsilon_\circ = \min\{\varepsilon_1, \varepsilon_2\}$ we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_\circ\delta_{i_\circ}u_{\circ i_\circ}(t) \geq 0$, for all $t \in [a, b]$. This gives us that $Mu_\circ \geq \varepsilon_\circ u_\circ$ with respect to the cone $\mathcal{P}_1$. By applying Theorem 1.6, the conclusions of our theorem follow.

We have a parallel theorem in the case that $i_1 = 0$.

THEOREM 2.10. *Let $Ly = 0$ be right disfocal, and assume that $\delta_i\delta_j p_{ij}(t) \geq 0$, for $t \in [a, b]$, $1 \leq i, j \leq m$, and that there is a $t_\circ \in [a, b]$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$. Then for eigenvalue problem (1)*

$$(-1)^{n-1} Lu = \lambda P(t)u$$

$$T_0 u = 0, \quad (\textit{so } i_1 = 0),$$

there exists an eigenvector $z_\circ \in \mathcal{P}_0$ with corresponding positive eigenvalue $\lambda_\circ$ which is a lower bound for the modulus of any other eigenvalue for the corresponding problem.

PROOF: Like before, we solve this problem by seeking the eigenvalues of the linear integral operator $M : \mathcal{B}_0 \to \mathcal{B}_0$ defined by

$$Mu(t) = \int_a^b G(t, s)P(s)u(s)\, ds, \quad a \leq t \leq b,$$

where $G(t, s)$ is the Green's function for (2), under the boundary conditions $T_0 y = 0$. Again, the eigenvalues of the boundary value problem (1) are reciprocals of the eigenvalues of the operator M, and we note that zero is not an eigenvalue of (1) since $Ly = 0$ is assumed to be right disfocal.

Now, an argument identical to the one in the proof of Theorem 2.9, shows that our compact operator M maps $\mathcal{P}_0$ into $\mathcal{P}_0$.

In order to apply Theorem 1.6, we must find a nontrivial $u_\circ \in \mathcal{P}_0$, and an $\varepsilon_\circ > 0$ so that $Mu_\circ \geq \varepsilon_\circ u_\circ$. In this case we let

$$u_\circ(t) = (-1)^{k_\circ} \frac{(t-a)^k}{k!} \frac{(t-b)^{k_\circ}}{k_\circ!} \delta_{i_\circ} e_{i_\circ},$$

where $e_{i_\circ}$ is the unit vector in $\mathcal{R}^m$ in the $i_\circ$ direction. We note that $u \in \mathcal{B}_0$. It is easy to see that δ_j times the $j\,th$ component of $u_\circ(t)$ is nonnegative. Hence $u_\circ \in \mathcal{P}_0$. We also have that $\delta_{i_\circ} u_{\circ i_\circ}(t) > 0$ on (a,b) and that $\delta_{i_\circ} u_{\circ i_\circ}^{(k)}(a) = (-1)^{k_\circ} \frac{(a-b)^{k_\circ}}{k_\circ!} > 0$, and $(-1)^{k_\circ} \delta_{i_\circ} u_{\circ i_\circ}^{(k)}(b) = \frac{(b-a)^k}{k!} > 0$.

We now consider $Mu_\circ(t)$. Since $M\colon \mathcal{P}_0 \to \mathcal{P}_0$, we know that $\delta_j(Mu_\circ)_j(t) \geq 0 = \delta_j u_{\circ j}(t)$ for $1 \leq j \leq m$, $j \neq i_\circ$. When $j = i_\circ$ we have that

$$\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) = \int_a^b G(t,s) p_{i_\circ i_\circ}(s) (-1)^{k_\circ} \frac{(t-a)^k}{k!} \frac{(t-b)^{k_\circ}}{k_\circ!}\, ds > 0,$$

for $t \in (a,b)$ since by Theorems 2.2 and 2.6, $G(t,s) > 0$ for $t \in (a,b)$, $s \in (a,b)$, and $p_{i_\circ i_\circ}(t_\circ) > 0$, $p_{i_\circ i_\circ}$ continuous. So we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) > 0$ for all $t \in (a,b)$.

Now similar to the proof of Theorem 2.9, we can find an $\varepsilon_1 > 0$, and a $\delta_1 > 0$, so that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_1 \delta_{i_\circ} u_{\circ i_\circ}(t) \geq 0$, for all $t \in [a, a+\delta]$.

Also, Theorem 2.6 tells us that $(-1)^{k_\circ} G^{(k_\circ)}(b,s) > 0$ for all $s \in (a,b)$, so we

can see from above that

$$(-1)^{k_\circ}\delta_{i_\circ}(Mu_\circ)_{i_\circ}^{(k_\circ)}(b) = \int_a^b (-1)^{k_\circ} G^{(k_\circ)}(b,s)p_{i_\circ i_\circ}(s)(-1)^{k_\circ}\frac{(t-a)^k}{k!}\frac{(t-b)^{k_\circ}}{k_\circ!}\,ds$$
$$> 0.$$

Thus, there exists an $\varepsilon_2 > 0$ so that $(-1)^{k_\circ}\delta_{i_\circ}(Mu_\circ)_{i_\circ}^{(k)}(b) - \varepsilon_2(-1)^{k_\circ}\delta_{i_\circ}u_{\circ i_\circ}^{(k)}(b) > 0$. Now $(-1)^{k_\circ}\delta_{i_\circ}(Mu_\circ)_{i_\circ}^{(j)}(b) - \varepsilon_2(-1)^{k_\circ}\delta_{i_\circ}u_{\circ i_\circ}^{(j)}(b) = 0$, for $j = 0, 1, \ldots, k_\circ - 1$. Thus by using a Taylor series expansion, we can find a $\delta_2 > 0$ so that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_2\delta_{i_\circ}u_{\circ i_\circ}(t) \geq 0$, for all $t \in [b-\delta_2, b]$.

Now both $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t)$ and $\delta_{i_\circ}u_{\circ i_\circ}(t)$ are positive on $[a+\delta_1, b-\delta_2]$ so as in the proof of the last theorem, we can find an $\varepsilon_3 > 0$ so that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_2\delta_{i_\circ}u_{\circ i_\circ}(t) \geq 0$, for all $t \in [a+\delta_1, b-\delta_2]$.

Finally, letting $\varepsilon_\circ = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_\circ\delta_{i_\circ}u_{\circ i_\circ}(t) \geq 0$, for all $t \in [a, b]$. This gives us that $Mu_\circ \geq \varepsilon_\circ u_\circ$ with respect to the cone $\mathcal{P}_0$. By applying Theorem 1.6, the conclusions of our theorem follow.

If we have stronger conditions on $P(t)$, we get better results. Again we will have parallel theorems.

THEOREM 2.11. *Let $Ly = 0$ be right disfocal on $[a, b]$ and assume $\delta_i\delta_j p_{ij}(t) \geq 0$, $1 \leq i, j \leq m$, for all $t \in [a, b]$, and p_{ij} equals zero only on a set of measure zero.*

Then for the eigenvalue problem (1),

$$(-1)^{n-1}\,Lu = \lambda P(t)u$$

$$T_1 u = 0, \quad (\textit{so } i_1 > 0),$$

there exists an essentially unique eigenvector $z_\circ$ in $\mathcal{P}_1^\circ$, and its corresponding eigenvalue is simple, positive and smaller then the modulus of any other eigenvalue for this eigenvalue problem.

PROOF: As in the last proof, we define the compact linear integral operator M by

$$Mu(t) = \int_a^b G(t,s)P(s)u(s)\,ds,$$

where $G(t,s)$ is the Green's function for (2). We wish to show that M is a $u_\circ$-positive operator so that we can apply Theorem 1.8. To do this, we will show that $M\colon \mathcal{P}_1\backslash\{0\} \to \mathcal{P}_1^\circ$ and then apply Lemma 1.5.

Let u be an arbitrary element in $\mathcal{P}_1\backslash\{0\}$. Then, there exists an $i_\circ \in \{1,2,\ldots,m\}$ and a $t_\circ \in (a,b)$, so that $\delta_{i\circ}u_{i_\circ}(t_\circ) > 0$. (By the continuity of $u_{i_\circ}$ we can assume, without loss of generality, that $t_\circ \in (a,b)$.) Since $u_{i_\circ}$ is a continuous function we have that there exists an interval to the right of $t_\circ$ on which $\delta_{i_\circ}u_{i\circ}$ is positive.

Now for each $i = 1,2,\ldots,m$, $\delta_i\delta_{i_\circ}p_{ii_\circ} \geq 0$, $p_{ii_\circ}$ is continuous and zero only on a set of measure zero. Thus, for each i, we can find an interval to the right of $t_\circ$, on which each $\delta_i\delta_{i_\circ}p_{ii_\circ}$ is positive. Taking the intersection of these $m+1$ right

intervals, we have an interval $(\alpha,\beta)\subset[a,b]$ such that $\delta_i\delta_{i_\circ}p_{ii_\circ}(t)\delta_{i_\circ}u_{i_\circ}(t)>0$, for all $t\in(\alpha,\beta)$, $i=1,2,\ldots,m$. Thus, since $G(t,s)>0$ for all $t\in(a,b]$, $s\in(a,b)$ by Theorem 2.2 and Theorem 2.6, and since by hypothesis $\delta_i\delta_{i_\circ}p_{ii_\circ}\geq 0$, we have that for each $i=1,2,\ldots,m$,

$$\begin{aligned}
\delta_i(Mu)_i(t) &= \int_a^b G(t,s)\delta_i\sum_{j=1}^m p_{ij}(s)u_j(s)\,ds\\
&= \int_a^b G(t,s)\sum_{j=1}^m \delta_i\delta_j p_{ij}(s)\delta_j u_j(s)\,ds\\
&\geq \int_\alpha^\beta G(t,s)\delta_i\delta_{i_\circ}p_{ii_\circ}(s)\delta_{i_\circ}u_{i_\circ}(s)\,ds\\
&> 0.
\end{aligned}$$

Thus we have that $\delta_i(Mu)_i(t)>0$ for all $t\in(a,b]$. But this gives us that $Mu(t)\in\mathcal{K}^\circ$ for all $t\in(a,b]$.

Now we also know by Theorem 2.6 that $G^{(k)}(a,s)>0$ for all $s\in(a,b)$. Following the same argument as above, this gives us that $(Mu)^{(k)}(a)\in\mathcal{K}^\circ$. Since $Mu(t)\in\mathcal{K}^\circ$ for all $t\in(a,b]$ and $(Mu)^{(k)}(a)\in\mathcal{K}^\circ$ we have by Lemma 2.7 that $Mu\in\mathcal{P}_1^\circ$. Now u was an arbitrary nontrivial element of $\mathcal{P}_1$. Thus we have that $M\colon\mathcal{P}_1\backslash\{0\}\to\mathcal{P}_1^\circ$. So by Lemma 1.5, we have that M is a $u_\circ$-positive operator. Hence we can now apply Theorem 1.8, and the conclusions of our theorem follow.

If we have that $i_1=0$ then we have results similar to Theorem 2.11.

THEOREM 2.12. *Let $Ly=0$ be right disfocal on $[a,b]$ and assume that $\delta_i\delta_j p_{ij}(t)\geq 0$, $1\leq i,j\leq m$, for all $t\in[a,b]$, and p_{ij} equals zero only on a set of measure zero.*

Then for the eigenvalue problem (1),

$$(-1)^{n-1}\, Lu = \lambda P(t)u$$

$$T_0 u = 0, \quad (so\ i_1 = 0),$$

there exists an essentially unique eigenvector $z_\circ$ *in* $\mathcal{P}_0^\circ$, *and its corresponding eigenvalue is simple, positive and smaller then the modulus of any other eigenvalue for this eigenvalue problem.*

PROOF: As in the last proof, we define the compact linear integral operator M by

$$Mu(t) = \int_a^b G(t,s)P(s)u(s)\, ds,$$

where $G(t,s)$ is the Green's function for (2), with boundary conditions $T_0 y = 0$. We will again show that M is a $u_\circ$-positive operator and then apply Theorem 1.8. To do this, we again show that $M\colon \mathcal{P}_0\backslash\{0\} \to \mathcal{P}_0^\circ$ and then apply Lemma 1.5.

Let u be an arbitrary element in $\mathcal{P}_0\backslash\{0\}$. Then following arguments identical to those in the last Theorem, we have that there exists an interval $(\alpha,\beta) \subset [a,b]$ and an $i_\circ$ so that $\delta_i \delta_{i_\circ} p_{ii_\circ}(t) \delta_{i_\circ} u_{i_\circ}(t) > 0$, for all $t \in (\alpha,\beta)$, $\quad i = 1,2,\ldots,m$. Then, since $G(t,s) > 0$ for all $t \in (a,b)$, $s \in (a,b)$ by Theorem 2.2 and Theorem

2.6, and since by hypothesis $\delta_i\delta_{i_\circ}p_{ii_\circ} \geq 0$, we have that for each $i = 1, 2, \ldots, m$,

$$\begin{aligned} \delta_i(Mu)_i(t) &= \int_a^b G(t,s)\delta_i \sum_{j=1}^m p_{ij}(s)u_j(s)\,ds \\ &= \int_a^b G(t,s) \sum_{j=1}^m \delta_i\delta_j p_{ij}(s)\delta_j u_j(s)\,ds \\ &\geq \int_\alpha^\beta G(t,s)\delta_i\delta_{i_\circ}p_{ii_\circ}(s)\delta_{i_\circ}u_{i_\circ}(s)\,ds \\ &> 0. \end{aligned}$$

Thus we have that $\delta_i(Mu)_i(t) > 0$ for all $t \in (a,b)$. But this gives us that $Mu(t) \in \mathcal{K}^\circ$ for all $t \in (a,b)$.

Now we also know by Theorem 2.6 that $G^{(k)}(a,s) > 0$ for all $s \in (a,b)$. Following the same argument as above, this gives us that $(Mu)^{(k)}(a) \in \mathcal{K}^\circ$. Theorem 2.6 also tells us that $(-1)^{k_\circ}G^{(k_\circ)}(b,s) > 0$ for all $s \in (a,b)$. Hence, similar to the argument above, we have that

$$(-1)^{k_\circ}\delta_i(Mu)_i^{(k_\circ)}(b) > 0,$$

and so $(-1)^{k_\circ}(Mu)^{(k_\circ)}(b) \in \mathcal{K}^\circ$.

Since $Mu(t) \in \mathcal{K}^\circ$ for all $t \in (a,b)$, $(Mu)^{(k)}(a) \in \mathcal{K}^\circ$ and $(-1)^{k_\circ}(Mu)^{(k_\circ)}(b) \in \mathcal{K}^\circ$ we have by Lemma 2.8 that $Mu \in \mathcal{P}_0^\circ$. Now u was an arbitrary nontrivial element of $\mathcal{P}_0$. Thus we have that $M\colon \mathcal{P}_0\backslash\{0\} \to \mathcal{P}_0^\circ$. So by Lemma 1.5, we have that M is a $u_\circ$-positive operator. Hence we can now apply Theorem 1.8, and the conclusions of this theorem follow.

We have now come to our main theorems which give comparison results for eigenvalue problems with different boundary conditions. The boundary conditions we will consider pertain to the n-tuples $(i_1, i_1, \ldots, i_{n-k})$ and $(j_1, j_2, \ldots, j_{n-k})$. We let $Ty = 0$ denote the boundary conditions $y^{(i)}(a) = 0$ for $i = 0, 1, \ldots, k-1$, and $y^{(i_j)}(b) = 0$ for $j = 1, 2, \ldots, n-k$. Also we let $Sy = 0$ denote the boundary conditions $y^{(i)}(a) = 0$ for $i = 0, 1, \ldots, k-1$, and $y^{(j_i)}(b) = 0$ for $i = 1, 2, \ldots, n-k$.

THEOREM 2.13. *Let $Ly = 0$ be right disfocal and assume that the continuous matrix function $P(t)$ and $Q(t)$ have the properties:*

a) *There is an $i_\circ \in \{1, 2, \ldots, m\}$ and a $t_\circ \in [a, b]$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$;*

b) *$0 \le \delta_i \delta_j p_{ij}(t) \le \delta_i \delta_j q_{ij}(t)$, for $t \in [a, b]$, $1 \le i, j \le m$;*

c) *Each $q_{ij} = 0$ only on a set of measure zero.*

Further assume that $(i_1, i_1, \ldots, i_{n-k}) < (j_1, j_2, \ldots, j_{n-k})$ and that $i_1 > 0$. Then there exists smallest positive eigenvalues $\lambda_\circ, \Lambda_\circ$ of (1) and (3), respectively,

$$(-1)^{n-1} Lu = \lambda P(t) u \qquad (-1)^{n-1} Lu = \Lambda Q(t) u$$

$$T_1 u = 0 \qquad\qquad S_1 u = 0.$$

both of which are positive, $\lambda_\circ$ a lower bound in modulus and $\Lambda_\circ$ strictly less in modulus then any other eigenvalue for their corresponding problems, and both of their corresponding eigenvectors belong to $\mathcal{P}_1$. Further, $\Lambda_\circ$ is a simple eigenvalue and its corresponding eigenvector belongs to $\mathcal{P}_1^\circ$. Moreover, $\Lambda_\circ \le \lambda_\circ$ and if $\lambda_\circ = \Lambda_\circ$, then $P(t) = Q(t)$ on $[a, b]$.

PROOF: Let $G(t, s)$ be the Green's function for $Ly = 0$, $T_1 y = 0$ and $H(t, s)$ be the Green's function for $Ly = 0$, $S_1 y = 0$. We define the integral operators

$M, N\colon \mathcal{B}_1 \to \mathcal{B}_1$ by

$$Mu(t) = \int_a^b G(t,s)P(s)u(s)\,ds \quad \text{and} \quad Nu(t) = \int_a^b H(t,s)Q(s)u(s)\,ds.$$

By Theorem 2.2, we have that $0 < G(t,s) < H(t,s)$ on $(a,b)^2$. So from earlier proofs, we know that $M, N\colon \mathcal{P}_1 \to \mathcal{P}_1$. Now by Theorem 2.9, M possesses a positive eigenvalue $1/\lambda_\circ$ which is an upper bound, in modulus, for all other eigenvalues of M, and its corresponding eigenvector $z_\circ$ belongs to $\mathcal{P}_1$. By Theorem 2.11, we have that N has a positive, simple eigenvalue $1/\Lambda_\circ$, which is strictly greater, in modulus, than all other eigenvalues of N, and its essentially unique eigenvector $v_\circ$ belongs to $\mathcal{P}_1^\circ$.

To get a comparison between these two eigenvalues we need to show that $M \le N$, with respect to $\mathcal{P}_1$. Let u be an arbitrary element in $\mathcal{P}_1$. Then for each fixed $i \in \{1, 2, \ldots, m\}$, we have from the hypothesis,

$$\delta_i\delta_j q_{ij}(t) \ge \delta_i\delta_j p_{ij}(t)) \ge 0 \quad \text{for } t \in [a,b], \quad 1 \le j \le m.$$

Since $u \in \mathcal{P}_1$, we know that $\delta_j u_j(t) \ge 0$ for all $t \in [a,b]$, $1 \le j \le m$. This gives us that

$$\sum_{j=1}^m \delta_i q_{ij}(t)u_j(t) \ge \sum_{j=1}^m \delta_i p_{ij}(t)u_j(t) \ge 0$$

for $t \in [a,b]$, $1 \le j \le m$. Then from Theorem 2.2 we have that

$$\begin{aligned}
\int_a^b H(t,s)\sum_{j=1}^m \delta_i q_{ij}(s)u_j(s)\,ds \quad &\ge \quad \int_a^b G(t,s)\sum_{j=1}^m \delta_i p_{ij}(s)u_j(s)\,ds \quad \ge \quad 0\\
\delta_i\Big(\int_a^b H(t,s)\sum_{j=1}^m q_{ij}(s)u_j(s)\,ds\Big) \quad &\ge \quad \delta_i\Big(\int_a^b G(t,s)\sum_{j=1}^m p_{ij}(s)u_j(s)\,ds\Big) \quad \ge \quad 0.
\end{aligned}$$

Since i was arbitrary, this tells us that component wise,

$$\delta_i\Big(\int_a^b H(t,s)Q(s)u(s)\,ds\Big)_i \quad \geq \quad \delta_i\Big(\int_a^b G(t,s)P(s)u(s)\,ds\Big)_i \quad \geq \quad 0$$

for all $t \in [a,b]$, $i = 1, 2, \ldots, m$. Thus,

$$\Big(\int_a^b H(t,s)Q(s)u(s)\,ds - \int_a^b G(t,s)P(s)u(s)\,ds\Big) = (N-M)u(t) \in \mathcal{K}$$

for all $t \in [a,b]$. Thus $Nu \geq Mu$ with respect to the cone $\mathcal{P}_1$. Since u was an arbitrary element of $\mathcal{P}_1$, we have that $M \leq N$.

Now $(\frac{1}{\lambda_\circ}, z_\circ)$ and $(\frac{1}{\Lambda_\circ}, v_\circ)$ are eigenpairs of M and N respectively, so we have that the inequalities of Theorem 1.11 hold. Also, similair to the proof in Theorem 8, we have that N is $u_\circ$-positive. From above we have that $M \leq N$, and so we can apply Theorem 1.11 to give us that $\frac{1}{\lambda_\circ} \leq \frac{1}{\Lambda_\circ}$ or $\Lambda_\circ \leq \lambda_\circ$.

Finally, suppose that $\lambda_\circ = \Lambda_\circ \doteq \lambda$, then Theorem 1.11 tells us that $z_\circ = kv_\circ$ for some nonzero scalar k. Then $\lambda P(t)z_\circ = Lz_\circ = kLv_\circ = k\lambda Q(t)v_\circ = \lambda Q(t)z_\circ$. Thus $\lambda P(t)z_\circ = \lambda Q(t)z_\circ$ or $(Q(t) - P(t))z_\circ = 0$ since $\lambda \neq 0$. Comparing each component i of $(Q(t) - P(t))z_\circ$, gives us that

$$\sum_{j=1}^{m}(q_{ij}(t) - p_{ij}(t))z_{\circ j}(t) = 0, \qquad t \in [a,b].$$

So that

$$\sum_{j=1}^{m}[\delta_i\delta_j(q_{ij}(t) - p_{ij}(t))]\delta_j z_{\circ j}(t) = 0, \qquad t \in [a,b].$$

Since $z_\circ \in \mathcal{P}_1^\circ$ we have that $\delta_j z_\circ(t) > 0$ for all $t \in (a,b]$. This plus the fact that $\delta_i\delta_j q_{ij}(t) \geq \delta_i\delta_j p_{ij}(t) \geq 0$ for $t \in [a,b]$, $1 \leq i,j \leq m$, gives us

$$p_{ij}(t) = q_{ij}(t), \qquad t \in (a,b], \quad 1 \leq i,j \leq m.$$

Finally, by continuity it follows that $P(t) = Q(t)$ on the closed interval $[a, b]$.

Our companion theorem for Theorem 2.13, requires more of a correlation between the boundary conditions.

THEOREM 2.14. *Let $Ly = 0$ be right disfocal and assume that the continuous matrix function $P(t)$ and $Q(t)$ have the properties:*

a) *There is an $i_\circ \in \{1, 2, \ldots, m\}$ and a $t_\circ \in [a, b]$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$;*

b) $0 \le \delta_i \delta_j p_{ij}(t) \le \delta_i \delta_j q_{ij}(t), \quad$ *for $t \in [a, b]$,* $\quad 1 \le i, j \le m$;

c) *Each $q_{ij} = 0$ only on a set of measure zero.*

Further assume that $(i_1, i_1, \ldots, i_{n-k}) < (j_1, j_2, \ldots, j_{n-k})$, $i_1 = 0$ and that the integer $k_\circ$ defined for $(i_1, i_2, \ldots, i_{n-k})$ is also the $k_\circ$ defined for $(j_1, j_2, \ldots, j_{n-k})$. Then there exists smallest positive eigenvalues $\lambda_\circ, \Lambda_\circ$ of (1) and (3), respectively,

$$(-1)^{n-1}\, Lu = \lambda P(t) u \qquad (-1)^{n-1}\, Lu = \Lambda Q(t) u$$

$$T_0 u = 0 \qquad\qquad S_0 u = 0.$$

both of which are positive, $\lambda_\circ$ a lower bound in modulus and $\Lambda_\circ$ strictly less in modulus then any other eigenvalue for their corresponding problems, and both of their corresponding eigenvectors belong to $\mathcal{P}_0$. Further, $\Lambda_\circ$ is a simple eigenvalue and its corresponding eigenvector belongs to $\mathcal{P}_0^\circ$. Moreover, $\Lambda_\circ \le \lambda_\circ$ and if $\lambda_\circ = \Lambda_\circ$, then $P(t) = Q(t)$ on $[a, b]$.

PROOF: The proof for this theorem is virtually identical to the proof of the last theorem. We let $G(t, s)$ be the Green's function for $Ly = 0$, $T_0 y = 0$ and $H(t, s)$ be the Green's function for $Ly = 0$, $S_0 y = 0$. We define the integral operators

$M, N\colon \mathcal{B}_0 \to \mathcal{B}_0$ by

$$Mu(t) = \int_a^b G(t,s)P(s)u(s)\,ds \quad \text{and} \quad Nu(t) = \int_a^b H(t,s)Q(s)u(s)\,ds.$$

By Theorem 2.2, we have that $0 < G(t,s) < H(t,s)$ on $(a,b)^2$. So from earlier proofs, we know that $M, N\colon \mathcal{P}_0 \to \mathcal{P}_0$. Now by Theorem 2.10, M possesses a positive eigenvalue $1/\lambda_\circ$ which is an upper bound, in modulus, for all other eigenvalues of M, and its corresponding eigenvector $z_\circ$ belongs to $\mathcal{P}_0$. By Theorem 2.12, we have that N has a positive, simple eigenvalue $1/\Lambda_\circ$, which is strictly greater, in modulus, than all other eigenvalues of N, and its essentially unique eigenvector $v_\circ$ belongs to $\mathcal{P}_0^\circ$.

Now, the argument to show that $M \leq N$ with respect to the cone $\mathcal{P}_0$ is identical to the argument in Theorem 2.13. Thus, by applying Theorem 1.11 we have that $\Lambda_\circ \leq \lambda_\circ$. If $\Lambda_\circ = \lambda_\circ \doteq \lambda$, then by following the argument in Theorem 2.13, we see that $P(t) = Q(t)$ on $[a,b]$.

Comparison Theorems for Eigenvalue Problems for Right Disfocal Differential Equations

I) <u>INTRODUCTION</u>:

Let $n > 1$, $m \geq 1$ and define $Lu = u^{(n)} + p_1(t)u^{(n-1)} + \cdots + p_n(t)u$ where $u(t)$ is an m-column vector such that $u \in C^n([a,b], \mathcal{R}^m)$ and $p_i \in C[a,b]$, $i = 1, 2, \ldots, n$. Also let $P(t)$, $Q(t)$ be continuous $m \times m$ matrix functions on $[a,b]$ and let $t_1 < t_2 < \cdots < t_n$ where $t_i \in [a,b], i = 1, 2, \ldots, n$.

We consider the n-point right focal eigenvalue problem:

$$(1) \qquad (-1)^{n-1} Lu = \lambda P(t) u$$

$$Tu = 0$$

where $Tu = 0$ denotes the boundary conditions

$$u^{(i-1)}(t_i) = 0, \quad i = 1, 2, \ldots, n.$$

If $G(t,s)$ is the Green's function for the scalar boundary value problem,

$$(2) \qquad (-1)^{n-1} Ly = 0$$

$$Ty = 0,$$

where Ly and Ty are as above, but defined appropriately for the scalar case, then under certain sign conditions on $G(t,s)$ and conditions on $P(t)$ we can show the

existence of a smallest positive eigenvalue. And with further conditions on $P(t)$, that its corresponding eigenvector is essentially unique with respect to a 'cone'. We also have comparison results for the eigenvalue problems (1) and (3),

$$(3) \qquad (-1)^{n-1}Lu = \Lambda Q(t)u$$
$$Tu = 0.$$

Our results are new, even in the scalar case. Our technique will be to use sign conditions on the Green's function, appropriately define an integral operator and then apply the theory of u_o-positive operators with respect to a cone in a Banach space. The theory of operators on a cone, can be found in great detail in the books by Kranosel'skiĭ [9] and Deimling [2]. Related papers include those of Eloe and Henderson [3], Gentry and Travis [4], Hankerson and Peterson [5,6], Keener and Travis [7,8], Kreith [10], Schmitt and Smith [11], Tomastik [12,13], and Travis [14].

II) CONE THEORY:

The following will be a short review of the definitions and theorems we will be using for our results. This theory was developed in great detail by Krasnosel'skiĭ [9].

Let $\mathcal{B}$ be a Banach space and $\mathcal{P}$ a closed, non-empty subset of $\mathcal{B}$. We say that $\mathcal{P}$ is a *cone* provided that if $u, v \in \mathcal{P}$ then $\alpha u + \beta v \in \mathcal{P}$ for all $\alpha, \beta \geq 0$, and that if $-u, u \in \mathcal{P}$ then $u = 0$, the zero element of $\mathcal{B}$. We say that a cone $\mathcal{P}$ is

reproducing provided that $\mathcal{B} = \mathcal{P} - \mathcal{P} \doteq \{u - v | u, v \in \mathcal{P}\}$. A cone is called *solid* if it has a non-empty interior, $\mathcal{P}^\circ \neq \emptyset$.

The cone will induce a *partial ordering* on our space $\mathcal{B}$, if we write $u \leq v$ to mean that $v - u \in \mathcal{P}$. If M and N are operators on $\mathcal{B}$, then we will write $M \leq N$ (with respect to $\mathcal{P}$) provided that $Mu \leq Nu$ for all $u \in \mathcal{P}$. A linear operator M on $\mathcal{B}$, is called *positive* if $\mathcal{P}$ is invariant with respect to M, that is $M : \mathcal{P} \to \mathcal{P}$. The operator M is called $u_\circ$-positive provided $u_\circ \in \mathcal{P}$, and for every $u \in \mathcal{P}\backslash\{0\}$, there exists positive k_1, k_2 (generally depending on u) such that

$$k_1 u_\circ \leq Mu \leq k_2 u_\circ.$$

We will be using the following theorems, which can be found in [9].

THEOREM 1. *Let $\mathcal{B}$ be a Banach space and $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^\circ$, then M is $u_\circ$-positive with respect to $\mathcal{P}$.*

THEOREM 2. *Let $\mathcal{P}$ be a reproducing cone and M a linear compact positive operator. Assume there exists a nontrivial $u_\circ \in \mathcal{P}$ and an $\varepsilon_\circ > 0$ such that $Mu_\circ \geq \varepsilon_\circ u_\circ$. Then M has an eigenvector $z_\circ \in \mathcal{P}$ with corresponding eigenvalue $\lambda_\circ \geq \varepsilon_\circ$ and $\lambda_\circ$ is an upper bound for the moduli of the remaining eigenvalues of M.*

THEOREM 3. *Let $\mathcal{P}$ be a reproducing cone and M a linear compact $u_\circ$-positive operator. Then M has an essentially unique eigenvector in $\mathcal{P}$ and the corresponding*

eigenvalue is simple, positive and larger than the modulus of any other eigenvalue of M.

Our last theorem of this section is from Keener and Travis [7], and is a generalization of a result from Travis [14].

THEOREM 4. *Let M and N be linear operators of which one is u_o-positive. If $M \leq N$ and there exists $u_1, u_2 \in \mathcal{P}\backslash\{0\}$ and $\lambda_1, \lambda_2 > 0$ such that $Mu_1 \geq \lambda_1 u_1$ and $Nu_2 \leq \lambda_2 u_2$, then $\lambda_1 \leq \lambda_2$ and if $\lambda_1 = \lambda_2$ then u_1 is a scalar multiple of u_2.*

III) THE GREEN'S FUNCTION:

In this section we will give sufficient conditions for the existence and give an explicit form for the Green's function for our problem (2). We need the following definition.

Definition: The differential equation $Ly = 0$ is called *right disfocal* on an interval I if there does not exist a nontrivial solution y of $Ly = 0$ and points $t_1 \leq t_2 \leq \cdots \leq t_n \in I$ such that $y^{(i-1)}(t_i) = 0$ for $i = 1, 2, \ldots, n$.

We will also need to introduce some notation. For each fixed s in the interval $[t_1, t_n]$, let $\{y_o(t,s), y_1(t,s), \ldots, y_{n-1}(t,s)\}$ be the set of (linear independent) solutions of $Ly = 0$, where

$$y_k^{(j)}(t,s)|_{t=s} = \delta_{jk}, \qquad 0 \leq j, k \leq n-1,$$

and δ_{jk} is the Kronecker-delta function

$$\delta_{jk} = \begin{cases} 0, & \text{for } j \neq k \\ 1, & \text{for } j = k. \end{cases}$$

We can now give a theorem about our Green's function.

THEOREM 5. *Let $Ly = 0$ be right disfocal on $[a, b]$. Then the Green's function $G(t, s)$, for the right focal problem*

$$(-1)^{n-1} Ly = 0$$

$$Ty = 0$$

exists and is given by:

for $s \in [t_k, t_{k+1}]$, $\quad t \leq s$

$$G(t,s) = \frac{(-1)^{n-1}}{D} \begin{vmatrix} 0 & y_1(t,t_1) & \cdots & y_{n-1}(t,t_1) \\ 0 & y_1'(t_2,t_1) & \cdots & y_{n-1}'(t_2,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_1^{(k-1)}(t_k,t_1) & \cdots & y_{n-1}^{(k-1)}(t_k,t_1) \\ y_{n-1}^{(k)}(t_{k+1},s) & y_1^{(k)}(t_{k+1},t_1) & \cdots & y_{n-1}^{(k)}(t_{k+1},t_1) \\ y_{n-1}^{(k+1)}(t_{k+2},s) & y_1^{(k+1)}(t_{k+2},t_1) & \cdots & y_{n-1}^{(k+1)}(t_{k+2},t_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(n-1)}(t_n,s) & y_1^{(n-1)}(t_n,t_1) & \cdots & y_{n-1}^{(n-1)}(t_n,t_1) \end{vmatrix}$$

if $s \leq t$ then we replace the zero in the first row, first column by $y_{n-1}(t,s)$ with everything else remaining the same.

This holds for $k = 1, 2, \ldots, n-1$, and D is given by

$$D = \begin{vmatrix} y_1'(t_2,t_n) & y_2'(t_2,t_n) & \cdots & y_{n-1}'(t_2,t_n) \\ y_1''(t_3,t_n) & y_2''(t_3,t_n) & \cdots & y_{n-1}''(t_3,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_n,t_1) & y_2^{(n-1)}(t_n,t_1) & \cdots & y_{n-1}^{(n-1)}(t_n,t_1) \end{vmatrix}.$$

PROOF: To show that $G(t,s)$ is our Green's function for (1) we must show that $G(t,s)$ is well defined and that it satisfies the properties from Coppel [1]:

i) As a function of t, $G(t,s)$ satisfies $Ly = 0$ on $[t_1, s)$ and $(s, t_n]$;

ii) $TG(\cdot, s) = 0$ for each fixed s;

iii) As a function of t, $G(t,s)$ and its first $n-2$ derivatives are continuous at $t = s$, while $G^{(n-1)}(s^+, s) - G^{(n-1)}(s^-, s) = \frac{1}{(-1)^{n-1}} = (-1)^{n-1}$.

The Green's function, $G(t,s)$ is well defined provided that $D \neq 0$. To show this, we will assume that $D = 0$ and show this lead to a contradiction. Let $\mathbf{A}$ be the $(n-1)\times(n-1)$ matrix, $\mathbf{A} = (y_j^{(i)}(t_{i+1}, t_1))$, for $1 \leq i,j \leq n-1$, so we have that $|\mathbf{A}| = D$, where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$. Since $D = 0$ there exists a nontrivial column vector $\vec{\mathbf{C}} = (C_1, \ldots, C_{n-1})^T$ so that $\mathbf{A}\vec{\mathbf{C}} = \vec{0}$. Let $z(t) = C_1 y_1(t, t_1) + C_2 y_2(t, t_1) + \cdots + C_{n-1} y_{n-1}(t, t_1)$. Since $z(t)$ is a linear combination of solutions to $Ly = 0$, we have by the linearity of L that $Lz = 0$. Now $z(t_1) = 0$ since each $y_j(t_1, t_1) = 0$. Also $z^{(j)}(t_{j+1}) = 0$ for $j = 1, 2, \ldots, n-1$, since $z^{(j)}(t_{j+1})$ is the j-row of $\mathbf{A}$ times the column vector $\vec{\mathbf{C}}$ and $\mathbf{A}\vec{\mathbf{C}} = \vec{0}$. So $Lz = 0$, $Tz = 0$ and z is not identically equal to zero since $\vec{\mathbf{C}}$ is nontrivial. This contradicts $Ly = 0$ is right disfocal. Thus $D \neq 0$ and $G(t,s)$ is well defined. Now that we have established that $D \neq 0$, a standard argument using Taylor series will show that $D > 0$.

To prove the properties i)–ii), we first fix $s_\circ$ as an arbitrary element of $[t_1, t_n]$. Then to prove i), for $t < s_\circ$, we have $G(t, s_\circ) = d_2 y_1(t, t_1) + \cdots + d_n y_{n-1}(t, t_1)$

where the d_i's are constants which can be determined by expanding the determinant of $G(t, s_o)$ along the first row. Thus, in the variable t, $G(t, s_o)$ is a linear combination of solutions of $Ly = 0$, and so is itself a solution of $Ly = 0$ on $[t_1, s_o)$. Similarly, $G(t, s_o)$ is a solution of $Ly = 0$ on $(s_o, t_n]$.

To show that $G(t, s_o)$ satisfies the boundary conditions, we first note that $G(t_1, s_o) = 0$ since the first row of the determinant of $G(t_1, s_o)$ is all zeros. Also, we have that $G^{(k)}(t_{k+1}, s_o) = 0$, for $k = 1, \ldots, n-1$, since in this case the the first row and the $(k+1)^{st}$ row are equal, so the determinant is zero. Thus from the properties of determinants we have $TG(\cdot, s_o) = 0$, so ii) is proved.

To prove iii), let r and τ be variables where $r < s_o$ and $\tau > s_o$. Then, with

$s_\circ \in [t_k, t_{k+1}]$ we have $G(\tau, s_\circ) - G(r, s_\circ) =$

$$\frac{(-1)^{n-1}}{D}\begin{vmatrix} y_{n-1}(\tau,s_\circ) & y_1(\tau,t_1) & \dots & y_{n-1}(\tau,t_1) \\ 0 & y_1'(t_2,t_1) & \dots & y_{n-1}'(t_2,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_1^{(k-1)}(t_k,t_1) & \dots & y_{n-1}^{(k-1)}(t_k,t_1) \\ y_{n-1}^{(k)}(t_{k+1},s_\circ) & y_1^{(k)}(t_{k+1},t_1) & \dots & y_{n-1}^{(k)}(t_{k+1},t_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(n-1)}(t_n,s_\circ) & y_1^{(n-1)}(t_n,t_1) & \dots & y_{n-1}^{(n-1)}(t_n,t_1) \end{vmatrix}$$

$$- \frac{(-1)^{n-1}}{D}\begin{vmatrix} 0 & y_1(r,t_1) & \dots & y_{n-1}(r,t_1) \\ 0 & y_1'(t_2,t_1) & \dots & y_{n-1}'(t_2,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_1^{(k-1)}(t_k,t_1) & \dots & y_{n-1}^{(k-1)}(t_k,t_1) \\ y_{n-1}^{(k)}(t_{k+1},s_\circ) & y_1^{(k)}(t_{k+1},t_1) & \dots & y_{n-1}^{(k)}(t_{k+1},t_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(n-1)}(t_n,s_\circ) & y_1^{(n-1)}(t_n,t_1) & \dots & y_{n-1}^{(n-1)}(t_n,t_1) \end{vmatrix}$$

$$= \frac{(-1)^{n-1}}{D}\begin{vmatrix} y_{n-1}(\tau,s_\circ) & y_1(\tau,t_1) - y_1(r,t_1) & \dots & y_{n-1}(\tau,t_1) - y_{n-1}(r,t_1) \\ 0 & y_1'(t_2,t_1) & \dots & y_{n-1}'(t_2,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_1^{(k-1)}(t_k,t_1) & \dots & y_{n-1}^{(k-1)}(t_k,t_1) \\ y_{n-1}^{(k)}(t_{k+1},s_\circ) & y_1^{(k)}(t_{k+1},t_1) & \dots & y_{n-1}^{(k)}(t_{k+1},t_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(n-1)}(t_n,s_\circ) & y_1^{(n-1)}(t_n,t_1) & \dots & y_{n-1}^{(n-1)}(t_n,t_1) \end{vmatrix}.$$

Now $\lim_{\tau,r\to s_\circ}\{y_j^{(i)}(\tau,t_1) - y_j^{(i)}(r,t_1)\} = 0$ for $j = 1,2,\dots,n-1; i = 0,1,\dots,n-1$. Also $\lim_{\tau\to s_\circ} y_{n-1}^{(i)}(\tau,s_\circ) = 0$ for $i = 0,1,\dots,n-2$. Thus $G^{(i)}(s_\circ^+,s_\circ) - G^{(i)}(s_\circ^-,s_\circ) = 0$ for $i = 0,1,\dots,n-2$, so we have that $G(t,s_\circ)$ and its first $n-2$ derivatives are continuous at $t = s_\circ$. Finally, since $\lim_{\tau\to s_\circ} y_{n-1}^{(n-1)}(\tau,s_\circ) = 1$, we

have $G^{(n-1)}(s_\circ^+, s_\circ) - G^{(n-1)}(s_\circ^-, s_\circ) =$

$$\frac{(-1)^{n-1}}{D}\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & y_1'(t_2,t_1) & \cdots & y_{n-1}'(t_2,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_1^{(k-1)}(t_k,t_1) & \cdots & y_{n-1}^{(k-1)}(t_k,t_1) \\ y_{n-1}^{(k)}(t_{k+1},s_\circ) & y_1^{(k)}(t_{k+1},t_1) & \cdots & y_{n-1}^{(k)}(t_{k+1},t_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1}^{(n-1)}(t_n,s_\circ) & y_1^{(n-1)}(t_n,t_1) & \cdots & y_{n-1}^{(n-1)}(t_n,t_1) \end{vmatrix}$$

$$= \frac{(-1)^{n-1}}{D}\begin{vmatrix} y_1'(t_2,t_n) & y_2'(t_2,t_n) & \cdots & y_{n-1}'(t_2,t_n) \\ y_1''(t_3,t_n) & y_2''(t_3,t_n) & \cdots & y_{n-1}''(t_3,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_n,t_1) & y_2^{(n-1)}(t_n,t_1) & \cdots & y_{n-1}^{(n-1)}(t_n,t_1) \end{vmatrix}$$

$$= \frac{(-1)^{n-1}}{D} \quad D.$$

So $G^{(n-1)}(s_\circ^+, s_\circ) - G^{(n-1)}(s_\circ^-, s_\circ) = (-1)^{n-1}$ and condition iii) is satisfied.

Since $s_\circ$ was an arbitrary element of $[t_1, t_n]$, we have that $G(t,s)$ is the Green's function for (2).

We close this section with the following hypothesis:

HYPOTHESIS (H). *Let $Ly = 0$ be right disfocal on $[a,b]$. We assume that the Green's function for (2), has the following properties:*

i) $G(t,s) > 0$ *for* $t \in (t_1, t_n]$, $s \in (t_1, t_n)$;

ii) $G'(t_1, s) > 0$ *for* $s \in (t_1, t_n)$.

This hypothesis is not true in all cases, but we will show sufficient conditions for (H) to hold for $n = 2, 3$ and 4.

IV) EXISTENCE AND COMPARISON RESULTS:

We will now introduce a suitable Banach space for our eigenvalue problem (1). Let

$$\mathcal{B} = \{u \in C^n([t_1, t_n], \mathcal{R}^m) \,|\, u(t_1) = 0\}$$

with norm $||u|| = \max_{0 \le i \le n}\{\max_{[t_1,t_n]} |u^{(i)}(t)|\}$ where $|\cdot|$ is the Euclidean norm. Following ideas from Hankerson and Peterson [5,6], and Tomastik's paper [13], we let $I, J \subseteq \{1, 2, \ldots, m\}$ be such that $I \cup J = \{1, 2, \ldots, m\}$ and $I \cap J = \emptyset$. (It is permissible for $I = \emptyset$ or $J = \emptyset$.) Let $\mathcal{K}$ be the 'quadrant' cone in $\mathcal{R}^m$ defined by

$$\mathcal{K} = \{x = (x_1, \ldots, x_m) \,|\, x_i \ge 0 \text{ if } i \in I,\; x_i \le 0 \text{ if } i \in J\}.$$

Although some of our results will hold for any solid cone in $\mathcal{R}^m$, we will just concern ourselves with $\mathcal{K}$ being a 'quadrant' cone in $\mathcal{R}^m$. Define δ_i to be the discrete function $\delta_i = 1$ if $i \in I$, and $\delta_i = -1$ if $i \in J$. We can then equivalently define the cone $\mathcal{K}$ to be $\mathcal{K} = \{x \in \mathcal{R}^m \,|\, \delta_i x_i \ge 0 \text{ for } i = 1, 2, \ldots, m\}$. This also allows us to define the interior of $\mathcal{K}$ as $\mathcal{K}^\circ = \{x \in \mathcal{R}^m \,|\, \delta_i x_i > 0 \text{ for } i = 1, 2, \ldots, m\}$.

We now define the reproducing cone $\mathcal{P} \subset \mathcal{B}$ by $\mathcal{P} = \{u \in \mathcal{B} \,|\, u(t) \in \mathcal{K},\, t \in [t_1, t_n]\}$. This gives us the following Lemma concerning the interior of our cone $\mathcal{P}$.

LEMMA 6. *Let the cone $\mathcal{P}$ in the Banach space $\mathcal{B}$ be defined as above. Then the interior of $\mathcal{P}$ is given by*

$$\mathcal{P}^\circ = \{u \in \mathcal{B} \,|\, u(t) \in \mathcal{K}^\circ,\, t \in (t_1, t_n] \text{ and } u'(t_1) \in \mathcal{K}^\circ\}.$$

PROOF: Let $Q = \{u \in \mathcal{B} \,|\, u(t) \in \mathcal{K}^\circ,\ t \in (t_1, t_n] \text{ and } u'(t_1) \in \mathcal{K}^\circ\}$. First we will show that $Q \subseteq \mathcal{P}^\circ$. Let u be an arbitrary element of Q, so we want to find an $\varepsilon > 0$ so that the ball $B(u;\varepsilon) \subset \mathcal{P}$. For a vector function $x(t)$ on $[\alpha,\beta] \subseteq [t_1, t_n]$ we define the distance function $d_{[\alpha,\beta]}(x(t), \partial\mathcal{K})$ to be the distance between the function $x(t)$ on $[\alpha,\beta]$ and the boundary of the cone $\partial\mathcal{K}$. Let $\varepsilon_1 = \frac{1}{2} d_{[t_1,t_n]}(u'(t_1), \partial\mathcal{K})$, so we have that $\varepsilon_1 > 0$ since $u'(t_1) \in \mathcal{K}^\circ$. Now u' is a continuous function, so there exists a $\delta > 0$ so that $u'(t) \in B(u'(t_1);\varepsilon_1) \subset \mathcal{R}^m$, for all $t \in [t_1, t_1 + \delta]$. We note that this gives us that $d_{[t_1,t_1+\delta]}(u'(t), \partial\mathcal{K}) > \varepsilon_1$.

We have that $u(t) \in \mathcal{K}^\circ$ for all $t \in [t_1 + \delta, t_n]$. Then, if we let ε_2 be $\varepsilon_2 = \frac{1}{2} d_{[t_1+\delta,t_n]}(u(t), \partial\mathcal{K})$ we also have that $\varepsilon_2 > 0$ since the graph of $u(t)$, which is compact on $[t_1 + \delta, t_n]$ and $\partial\mathcal{K}$ do not intersect. We note that in this case, we have that $d_{[t_1+\delta,t_n]}(u(t), \partial\mathcal{K}) > \varepsilon_2$.

Let $\varepsilon = min\{\varepsilon_1, \varepsilon_2\} > 0$. Then we have that $B(u;\varepsilon) \subset \mathcal{P}$. To show this, we let $z \in B(u;\varepsilon)$. Then $||z - u|| < \varepsilon$ so in particular we have that $|z'(t_1) - u'(t_1)| < \varepsilon_1 = \frac{1}{2} d_{[t_1,t_n]}(u'(t_1), \partial\mathcal{K})$. This tells us that $z'(t_1) \in \mathcal{K}^\circ$. Now $||z - u|| < \varepsilon$ also tells us that $|z'(t) - u'(t)| < \varepsilon$ for all $t \in [t_1, t_1 + \delta]$. This gives us that $z'(t) \in \mathcal{K}^\circ$ for all $t \in [t_1, t_1 + \delta]$. If this were not so, then since $z'(t_1) \in \mathcal{K}^\circ$ and z' is continuous, there would exists a $t_\circ \in [t_1, t_1 + \delta]$ so that $z'(t_\circ) \in \partial\mathcal{K}$. But from the note above we know that $d_{[t_1,t_1+\delta]}(u'(t), \partial\mathcal{K}) > \varepsilon_1 \geq \varepsilon$. This gives us that $|z'(t_\circ) - u'(t_\circ)| \geq \varepsilon$ which is a contradiction. Thus $z'(t) \in \mathcal{K}^\circ$ for all $t \in [t_1, t_1 + \delta]$. Now this in turn tells us that for $i = 1, 2, \ldots, m$, $\delta_i z_i'(t) > 0$ for all $t \in [t_1, t_1 + \delta]$. Thus $\delta_i z_i(t)$ is an

increasing function with $\delta_i z_i(t_1) = 0$ for each i. Hence we have that $\delta_i z_i(t) \geq 0$ for all $t \in [t_1, t_1 + \delta]$, for $i = 1, 2, \ldots, m$. That is, $z(t) \in \mathcal{K}$ for $t \in [t_1, t_1 + \delta]$.

Also, we have that $|z(t) - u(t)| < \varepsilon \leq \varepsilon_2$ for all $t \in [t_1 + \delta, t_n]$. Thus, $z(t) \notin \partial\mathcal{K}$ or else we contradict $d_{[t_1+\delta,t_n]}(u(t), \partial\mathcal{K}) > \varepsilon_2$. Since $z(t_1 + \delta) \in \mathcal{K}^\circ$ and z is continuous, we must have that $z(t) \in \mathcal{K}^\circ$ for all $t \in [t_1 + \delta, t_n]$

Thus $z(t) \in \mathcal{K}$ for all $t \in [t_1, t_n]$. But this means that $z \in \mathcal{P}$, and since z was an arbitrary element of $B(u; \varepsilon)$, we have that $B(u; \varepsilon) \subset \mathcal{P}$. But u was an arbitrary element of Q and we found an $\varepsilon > 0$ so that $B(u; \varepsilon) \subset \mathcal{P}$. Thus we have that $Q \subseteq \mathcal{P}^\circ$.

We now show that $\mathcal{P}^\circ \subseteq Q$. Let u be an arbitrary element of $\mathcal{P}^\circ$. Suppose there exists a $t_\circ \in (t_1, t_n]$ so that $u(t_\circ) \in \partial\mathcal{K}$. This give us that there exists a component of u, say $u_{i_\circ}$, so that $u_{i_\circ}(t_\circ) = 0$. Considering the scalar equation, $\delta_{i_\circ} u_{i_\circ}(t) > 0$, it can be seen that for any $\varepsilon > 0$, since $\delta_{i_\circ} u_{i_\circ}(t_\circ) = 0$, we can find a function $\delta_{i_\circ} z_{i_\circ}(t) \in B(\delta_{i_\circ} u_{i_\circ}; \varepsilon)$ so that $\delta_{i_\circ} z_{i_\circ}(t_\circ) < 0$. If we let the vector function $z(t)$ equal $u(t)$ in each component except in the $i_\circ$ slot, and then in that slot let $(z(t))_{i_\circ} = z_{i_\circ}(t)$, then $z \in B(u; \varepsilon)$. But $z(t_\circ) \notin \mathcal{K}$ since $\delta_{i_\circ} z_{i_\circ}(t_\circ) < 0$. Thus $z \notin \mathcal{P}$. Now z was based on $\varepsilon > 0$. Thus, for any $\varepsilon > 0$ we can find a $z \in B(u; \varepsilon)$ and $z \notin \mathcal{P}$. This contradicts $u \in \mathcal{P}^\circ$. Thus $u(t) \in \mathcal{K}^\circ$ for all $t \in (t_1, t_n]$.

Now suppose $u'(t_1) \notin \mathcal{K}^\circ$. So there exists an i so that $\delta_i u_i'(t_1) \leq 0$. Then for any $\varepsilon > 0$ we can find a $z \in B(u; \varepsilon)$ so that $\delta_i z_i'(t_1) < 0$. Thus $\delta_i z_i'$ is decreasing at t_1. We have that $z_i(t_1) = 0$ so we can find a $\delta > 0$ so that $\delta_i z_i(t) < 0$ for any

$t \in (t_1, t_1 + \delta]$. But this gives us that $z(t_\circ) \notin \mathcal{K}$ and so $z \notin \mathcal{P}$, which contradicts $u \in \mathcal{P}^\circ$. Thus we must have that $u'(t_1) \in \mathcal{K}^\circ$.

So if $u \in \mathcal{P}^\circ$ we have that $u(t) \in \mathcal{K}^\circ$ for all $t \in (t_1, t_n]$, and also that $u'(t_1) \in \mathcal{K}^\circ$. Thus $u \in Q$, and since u was an arbitrary element of $\mathcal{P}^\circ$, we have that $\mathcal{P}^\circ \subseteq Q$. Thus our lemma is proved.

With our Lemma out of the way, we can now proceed on to our first existence result.

THEOREM 7. *Assume hypothesis (H) holds, $\delta_i \delta_j p_{ij}(t) \geq 0$, for $t \in [t_1, t_n]$, $1 \leq i, j \leq m$, and that there is a $t_\circ \in [t_1, t_n]$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$. Then for eigenvalue problem (1)*

$$(-1)^{n-1} Lu = \lambda P(t) u$$

$$Tu = 0,$$

there exists an eigenvector $z_\circ \in \mathcal{P}$ with corresponding positive eigenvalue $\lambda_\circ$ which is a lower bound for the modulus of any other eigenvalue for the corresponding problem.

PROOF: To solve this problem, we will seek the eigenvalues of the linear integral operator $M: \mathcal{B} \to \mathcal{B}$ defined by

$$Mu(t) = \int_{t_1}^{t_n} G(t,s) P(s) u(s)\, ds, \quad t_1 \leq t \leq t_n,$$

where $G(t,s)$ is the Green's function for (2). Now the eigenvalues of the boundary

value problem (1) are reciprocals of the eigenvalues of the operator M. We note that zero is not an eigenvalue of (1) since $Ly = 0$ is assumed to be right disfocal.

Now an argument using the *Arzela-Ascoli* Theorem shows that M is a compact operator. We now show that $M : \mathcal{P} \to \mathcal{P}$. Let u be an arbitrary element of $\mathcal{P}$. If we have $\delta_i(Mu(t))_i \geq 0$ for all $t \in [t_1, t_n]$, $i = 1, 2, \ldots, m$, then $Mu \in \mathcal{P}$. Consider the ith component of $P(t)u(t)$, $\big(P(t)u(t)\big)_i = \sum_{j=1}^{m} p_{ij}(t)u_j(t)$. Now $\delta_j\delta_j = 1$, and $\delta_j u_j(t) \geq 0$ so we have that for all $t \in [t_1, t_n]$,

$$\delta_i(P(t)u(t))_i = \sum_{j=1}^{m} \delta_i\delta_j p_{ij}(t)\delta_j u_j(t) \geq 0,$$

since $\delta_i\delta_j p_{ij}(t) \geq 0$ by hypothesis. From Hypothesis (H), we have that $G(t,s) \geq 0$ for $t \in [t_1, t_n]$ and $s \in (t_1, t_n)$. Thus

$$\delta_i(Mu)_i(t) = \int_{t_1}^{t_n} G(t,s) \sum_{j=1}^{m} \delta_i\delta_j p_{ij}(s)\delta_j u_j(s)\, ds \geq 0,$$

for $t \in [t_1, t_n]$,$1 \leq i \leq m$, so $Mu \in \mathcal{P}$. Since u was an arbitrary element of $\mathcal{P}$, we have that M is a positive operator, that is $M : \mathcal{P} \to \mathcal{P}$.

In order to apply Theorem 2, we must find a nontrivial $u_\circ \in \mathcal{P}$, and an $\varepsilon_\circ > 0$ so that $Mu_\circ \geq \varepsilon_\circ u_\circ$. Let $u_\circ(t) = (t - t_1)\delta_{i_\circ} e_{i_\circ}$, where $e_{i_\circ}$ is the unit vector in $\mathcal{R}^m$ in the $i_\circ$ direction. This gives us that the jth component of $u_\circ(t)$, $u_{\circ j}(t) = (t - t_1)\delta_{i_\circ}\delta_{i_\circ j}$, where δ_{ij} is the Kronecker delta function. Thus $\delta_j u_{\circ j}(t) = \{\delta_j\delta_{i_\circ}(t - t_1)\}\delta_{i_\circ j} \geq 0$, so $u_\circ \in \mathcal{P}$. We note that $\delta_{i_\circ} u_{\circ i_\circ}(t) = (t - t_1) > 0$, on $(t_1, t_n]$ and that $\delta_{i_\circ} u'_{\circ i_\circ}(t_1) = 1 > 0$.

We now consider $Mu_\circ(t)$. Since $M\colon \mathcal{P} \to \mathcal{P}$, we know that $\delta_j(Mu_\circ)_j(t) \geq 0 = \delta_j u_{\circ j}(t)$ for $1 \leq j \leq m$, $j \neq i_\circ$. When $j = i_\circ$ we have that

$$\begin{aligned}
\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) &= \int_{t_1}^{t_n} G(t,s) \sum_{j=1}^{m} \delta_{i_\circ}\delta_j p_{i_\circ j}(s)\delta_j u_{\circ j}(s)\,ds \\
&= \int_{t_1}^{t_n} G(t,s)\delta_{i_\circ}\delta_{i_\circ} p_{i_\circ i_\circ}(s)\delta_{i_\circ} u_{\circ i_\circ}(s)\,ds \\
&= \int_{t_1}^{t_n} G(t,s) p_{i_\circ i_\circ}(s)(s - t_1)\,ds \\
&> 0, \quad \text{for } t \in (t_1, t_n],
\end{aligned}$$

since by hypothesis (H) $G(t,s) > 0$ for $t \in (t_1, t_n]$, $s \in (t_1, t_n)$, and $p_{i_\circ i_\circ}(t_\circ) > 0$, $p_{i_\circ i_\circ}$ continuous. So we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) > 0$ for all $t \in (t_1, t_n]$. Since again by hypothesis (H), $G'(t_1, s) > 0$ for all $s \in (t_1, t_n)$, we can see from above that $\delta_{i_\circ}(Mu_\circ)'_{i_\circ}(t_1) > 0$.

So for $\varepsilon_1 > 0$ sufficiently small, we have that $\delta_{i_\circ}(Mu_\circ)'_{i_\circ}(t_1) - \varepsilon_1\delta_{i_\circ} u'_{\circ i_\circ}(t_1) > 0$. Now $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t_1) - \varepsilon_1\delta_{i_\circ} u_{\circ i_\circ}(t_1) = 0$, so by continuity, there exists a $\delta > 0$ so that

$\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_1\delta_{i_\circ} u_{\circ i_\circ}(t) \geq 0$, for all $t \in [t_1, t_1 + \delta]$. Also, both $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t)$ and $\delta_{i_\circ} u_{\circ i_\circ}(t)$ are positive on $[t_1 + \delta, t_n]$ so we can let

$$\varepsilon_2 = \frac{\min_{[t_1+\delta, t_n]}(\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t)}{\max_{[t_1+\delta, t_n]}(\delta_{i_\circ} u_{\circ i_\circ}(t)} > 0.$$

This gives us that

$\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_2\delta_{i_\circ} u_{\circ i_\circ}(t) \geq 0$, for all $t \in [t_1 + \delta, t_n]$.

Finally, letting $\varepsilon_\circ = \min\{\varepsilon_1, \varepsilon_2\}$ we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) - \varepsilon_\circ\delta_{i_\circ} u_{\circ i_\circ}(t) \geq$

0, for all $t \in [t_1, t_n]$. This gives us that $Mu_\circ \geq \varepsilon_\circ u_\circ$ with respect to the cone $\mathcal{P}$. By applying Theorem 2, the conclusions of our theorem follow.

If we have stronger conditions on $P(t)$, we get better results.

THEOREM 8. *Assume hypothesis (H) holds, and $\delta_i \delta_j p_{ij}(t) \geq 0$, $1 \leq i, i \leq m$, for all $t \in [t_1, t_n]$, and p_{ij} equals zero only on a set of measure zero. Then for the eigenvalue problem (1),*

$$(-1)^{n-1} Lu = \lambda P(t) u$$

$$Tu = 0,$$

there exists an essentially unique eigenvector $z_\circ$ in $\mathcal{P}^\circ$, and its corresponding eigenvalue is simple, positive and smaller then the modulus of any other eigenvalue for this eigenvalue problem.

PROOF: As in the last proof, we define the compact linear integral operator M by

$$Mu(t) = \int_{t_1}^{t_n} G(t,s) P(s) u(s)\, ds.$$

We wish to show that M is a $u_\circ$-positive operator so that we can apply Theorem 3. To do this, we will show that $M\colon \mathcal{P}\backslash\{0\} \to \mathcal{P}$ and then apply Theorem 1.

Let u be an arbitrary element in $\mathcal{P}\backslash\{0\}$. Then, there exists a $t_\circ \in (t_1, t_n)$ and an $i_\circ \in \{1, 2, \ldots, m\}$ so that $\delta_{i_\circ} u_{i_\circ}(t_\circ) > 0$. (By the continuity of $u_{i_\circ}$ we can assume, without loss of generality, that $t_\circ \in (t_1, t_n)$.) Since $u_{i_\circ}$ is a continuous

function we have that there exists an interval to the right of t_o on which $\delta_{i_o} u_{i_o}$ is positive.

Now for each $i = 1, 2, \ldots, m$, $\delta_i \delta_{i_o} p_{ii_o} \geq 0$, p_{ii_o} is continuous and zero only on a set of measure zero. Thus, for each i, we can find an interval to the right of t_o, on which each $\delta_i \delta_{i_o} p_{ii_o}$ is positive. Taking the intersection of these $m+1$ right intervals, we have an interval $(\alpha, \beta) \subset [t_1, t_n]$ such that $\delta_i \delta_{i_o} p_{ii_o}(t) \delta_{i_o} u_{i_o}(t) > 0$, for all $t \in (\alpha, \beta)$, $i = 1, 2, \ldots, m$. Thus, since $G(t,s) > 0$ for all $t \in (t_1, t_n]$, $s \in (t_1, t_n)$ by hypothesis (H), and $\delta_i \delta_{i_o} p_{ii_o} \geq 0$, we have that for each $i = 1, 2, \ldots, m$,

$$\begin{aligned}
\delta_i (Mu)_i(t) &= \int_{t_1}^{t_n} G(t,s) \delta_i \sum_{j=1}^{m} p_{ij}(s) u_j(s)\, ds \\
&= \int_{t_1}^{t_n} G(t,s) \sum_{j=1}^{m} \delta_i \delta_j p_{ij}(s) \delta_j u_j(s)\, ds \\
&\geq \int_{\alpha}^{\beta} G(t,s) \delta_i \delta_{i_o} p_{ii_o}(s) \delta_{i_o} u_{i_o}(s)\, ds \\
&> 0.
\end{aligned}$$

Thus we have that $\delta_i (Mu)_i(t) > 0$ for all $t \in (t_1, t_n]$. But this gives us that $Mu(t) \in \mathcal{K}^\circ$ for all $t \in (t_1, t_n]$.

Now we also know by hypothesis (H) that $G'(t_1, s) > 0$ for all $s \in (t_1, t_n)$. Following the same argument as above, this gives us that $(Mu)'(t_1) \in \mathcal{K}^\circ$. Since $Mu(t) \in \mathcal{K}^\circ$ for all $t \in (t_1, t_n]$ and $(Mu)'(t_1) \in \mathcal{K}^\circ$ we have by Lemma 6 that $Mu \in \mathcal{P}^\circ$. Now u was an arbitrary nontrivial element of $\mathcal{P}$. Thus we have that $M\colon \mathcal{P} \backslash \{0\} \to \mathcal{P}^\circ$. So by Theorem 1, we have that M is a u_o-positive operator. Hence we can now apply Theorem 3, and the conclusions of our theorem follow.

We also have comparison results between two focal point eigenvalue problems.

THEOREM 9. *Assume hypothesis (H) holds and that the continuous matrix function $P(t)$ and $Q(t)$ have the properties:*

a) There is an $i_\circ \in \{1,2,\ldots,m\}$ and a $t_\circ \in [t_1,t_n]$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$;

b) $0 \le \delta_i\delta_j p_{ij}(t) \le \delta_i\delta_j q_{ij}(t)$, for $t \in [t_1,t_n]$, $1 \le i,j \le m$;

c) Each $q_{ij} = 0$ only on a set of measure zero.

Then there exists smallest positive eigenvalues $\lambda_\circ, \Lambda_\circ$ of (1) and (3),

$$(-1)^{n-1}\,Lu = \lambda P(t)u \qquad (-1)^{n-1}\,Lu = \Lambda Q(t)u$$

$$Tu = 0 \qquad\qquad\qquad Tu = 0.$$

both of which are positive, $\lambda_\circ$ a lower bound in modulus and $\Lambda_\circ$ strictly less in modulus then any other eigenvalue for their corresponding problems, and both of their corresponding eigenvectors belong to $\mathcal{P}$. Further, $\Lambda_\circ$ is a simple eigenvalue and its corresponding eigenvector belongs to $\mathcal{P}^\circ$. Moreover, $\Lambda_\circ \le \lambda_\circ$ and if $\lambda_\circ = \Lambda_\circ$, then $P(t) = Q(t)$ on $[t_1,t_n]$.

PROOF: We define the integral operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(t) = \int_{t_1}^{t_n} G(t,s)P(s)u(s)\,ds \quad \text{and} \quad Nu(t) = \int_{t_1}^{t_n} G(t,s)Q(s)u(s)\,ds,$$

where $G(t,s)$ is the Green's function for (2). We then have, from earlier proofs, that $M, N : \mathcal{P} \to \mathcal{P}$. Now by Theorem 7, M possesses a positive eigenvalue $1/\lambda_\circ$ which is an upper bound, in modulus, for all other eigenvalues of M, and its corresponding eigenvector $z_\circ$ belongs to $\mathcal{P}$. By Theorem 8, we have that N has

a positive, simple eigenvalue $1/\Lambda_o$, which is strictly greater, in modulus, than all other eigenvalues of N, and its essentially unique eigenvector v_o belongs to $\mathcal{P}^\circ$.

We will now show that $M \le N$, with respect to $\mathcal{P}$. Let u be an arbitrary element in $\mathcal{P}$. Then for each fixed $i \in \{1, 2, \ldots, m\}$, we have $\delta_i\delta_j(q_{ij}(t) - p_{ij}(t)) \ge 0$, for $t \in [t_1, t_n]$, $1 \le j \le m$. Since $u \in \mathcal{P}$, we know that $\delta_j u_j(t) \ge 0$ for all $t \in [t_1, t_n]$, $1 \le j \le m$. This gives us that

$$\sum_{j=1}^{m} \delta_i(q_{ij}(t) - p_{ij}(t))u_j(t) \ge 0$$

for $t \in [t_1, t_n]$, $1 \le j \le m$. Now hypothesis (H) tells us that $G(t,s) \ge 0$ on $(t_1, t_n)^2$. Thus

$$\int_{t_1}^{t_n} G(t,s) \sum_{j=1}^{m} \delta_i(q_{ij}(s) - p_{ij}(s))u_j(s)\,ds \ge 0$$

$$\delta_i\Big(\int_{t_1}^{t_n} G(t,s) \sum_{j=1}^{m}(q_{ij}(s) - p_{ij}(s))u_j(s)\,ds\Big) \ge 0.$$

Since i was arbitrary, each component of $\int_{t_1}^{t_n} G(t,s)(Q(s) - P(s))u(s)\,ds$ times δ_i is greater than or equal to zero for all $t \in [t_1, t_n]$. Thus, $\int_{t_1}^{t_n} G(t,s)(Q(s) - P(s))u(s)\,ds = (N - M)u(t) \in \mathcal{K}$ for all $t \in [t_1, t_n]$. Thus $Nu \ge Mu$ with respect to the cone $\mathcal{P}$. Since u was an arbitrary element of $\mathcal{P}$, we have that $M \le N$.

Now $(\frac{1}{\lambda_o}, z_o)$ and $(\frac{1}{\Lambda_o}, v_o)$ are eigenpairs of M and N respectively, so we have that the inequalities of Theorem 4 hold. Also, similair to the proof in Theorem 8, we have that N is u_o-positive. From above we have that $M \le N$, and so we can apply Theorem 4 to give us that $\frac{1}{\lambda_o} \le \frac{1}{\Lambda_o}$ or $\Lambda_o \le \lambda_o$.

Finally, suppose that $\lambda_\circ = \Lambda_\circ \doteq \lambda$, then Theorem 4 tells us that $z_\circ = kv_\circ$ for some nonzero scalar k. Then $\lambda P(t)z_\circ = Lz_\circ = kLv_\circ = k\lambda Q(t)v_\circ = \lambda Q(t)z_\circ$. Thus $\lambda P(t)z_\circ = \lambda Q(t)z_\circ$ or $(Q(t) - P(t))z_\circ = 0$ since $\lambda \neq 0$. Comparing each component i of $(Q(t) - P(t))z_\circ$, gives us that

$$\sum_{j=1}^{m}(q_{ij}(t) - p_{ij}(t))z_{\circ j}(t) = 0, \qquad t \in [t_1, t_n].$$

So that

$$\sum_{j=1}^{m}[\delta_i\delta_j(q_{ij}(t) - p_{ij}(t))]\delta_j z_{\circ j}(t) = 0, \qquad t \in [t_1, t_n].$$

Since $z_\circ \in \mathcal{P}^\circ$ we have that $\delta_j z_\circ(t) > 0$ for all $t \in (t_1, t_n]$. This plus the fact that $\delta_i\delta_j q_{ij}(t) \geq \delta_i\delta_j p_{ij}(t) \geq$ for $t \in [t_1, t_n]$, $1 \leq i, j \leq m$, gives us

$$p_{ij}(t) = q_{ij}(t), \qquad t \in (t_1, t_n], \quad 1 \leq i, j \leq m.$$

Finally, by continuity it follows that $P(t) = Q(t)$ on the closed interval $[t_1, t_n]$.

V) EXAMPLES

In our final section, we will give examples for which hypothesis (H) holds.

Example n=2:

In this example we have $Lu = u'' + p_1(t)u' + p_2(t)u$. Let t_1, t_2 be elements of any interval I over which L is right disfocal. Then, from Theorem 5, our Green's function for (2) is

$$G(t,s) = \begin{cases} \frac{-1}{y_1'(t_2,t_1)} \begin{vmatrix} 0 & y_1(t,t_1) \\ y_1'(t_2,s) & y_1'(t_2,t_1) \end{vmatrix} & t_1 \leq t \leq s \leq t_2, \\ \frac{-1}{y_1'(t_2,t_1)} \begin{vmatrix} y_1(t,s) & y_1(t,t_1) \\ y_1'(t_2,s) & y_1'(t_2,t_1) \end{vmatrix} & t_1 \leq s \leq t \leq t_2. \end{cases}$$

Now consider $y_1(t,s)$ for any $t,s \in [t_1,t_2]$. We know that $y_1(s,s) = 0$ and $y_1(t,s) \neq 0$ for all $t \neq s$ or else by Rolle's Theorem we contradict $Lu = 0$ is right disfocal. Thus $y_1(t,s) < 0$ for all $t < s$ and $y_1(t,s) > 0$ for all $t > s$. We also know that $y_1'(s,s) = 1$ and that $y_1'(t,s) \neq 0$ for all $t > s$ or else we again have a contradiction. Thus we have that $y_1'(t,s) > 0$ for all $t > s$.

Then, when $t_1 \leq t \leq s \leq t_2$ we have that

$$\begin{aligned} G(t,s) &= \frac{-1}{y_1'(t_2,t_1)}\{-y_1'(t_2,s)y_1(t,t_1)\} \\ &= \frac{y_1'(t_2,s)y_1(t,t_1)}{y_1'(t_2,t_1)}. \end{aligned}$$

So $G(t,s) \geq 0$ and positive when $t1 < t \leq s$.

When $t_1 < s \leq t \leq t_2$ we have

$$\begin{aligned} G(t,s) &= \frac{-1}{y_1'(t_2,t_1)}\{y_1(t,s)y_1'(t_2,t_1) - y_1'(t_2,s)y_1(t,t_1)\} \\ &= \frac{y_1'(t_2,s)y_1(t,t_1) - y_1(t,s)y_1'(t_2,t_1)}{y_1'(t_2,t_1)} > 0. \end{aligned}$$

Let $z(t) = y_1'(t_2,s)y_1(t,t_1) - y_1(t,s)y_1'(t_2,t_1)$. Then z is a solution of $Ly = 0$ and further, $z'(t_2) = 0$ and $z(s) > 0$ since we have shown that $G(t,s) > 0$ on $(t_1,s]$. Thus we must have that $z(t) > 0$ on $[s,t_2]$ or else we contradict $Ly = 0$ is right disfocal. Thus we have that $G(t,s) > 0$ on $[s,t_2]$ and hence $G(t,s) > 0$ for $t \in (t_1,t_2]$, $s \in (t_1,t_2)$.

Also we have that

$$\begin{aligned} G'(t_1,s) &= \frac{y_1'(t_2,s)y_1'(t_1,t_1)}{y_1'(t_2,t_1)} \\ &= \frac{y_1'(t_2,s)}{y_1'(t_2,t_1)} \\ &> 0, \qquad s \in (t_1,t_2). \end{aligned}$$

Thus we have that when $n = 2$, hypothesis (H) holds over any interval on which $Lu = 0$ is right disfocal.

In our next two examples we will take L to be $Ly = y^{(n)}$. We note here that when $Ly = y^{(n)}$, then $Ly = 0$ is right disfocal over any interval I. This can be seen by the fact that if y is a solution of $Ly = 0$, which satisfies $y^{(i-1)}(t_i) = 0$, for $i = 1, 2, \ldots, n$,then $y^{(n)}(t) = 0$, for all $t \in I$. This tells us that $y^{(n-1)}(t)$ is constant, but $y^{(n-1)}(t_n) = 0$. Thus $y^{(n-1)} = 0$ so $y^{(n-2)}$ is constant. But again $y^{(n-2)}(t_{n-1}) = 0$, so $y^{(n-2)} = 0$. Continuing in this manner we get that $y = 0$. Thus the only solution to $Ly = 0$ which satisfies the boundary contitions is the trivial solution, that is, $Ly = 0$ is right disfocal on I.

By taking $Ly = y^{(n)}$, our set of n linearly independent solutions to $Ly = 0$ is $\{1, (t-s), \ldots, (t-s)^{(n-1)}/(n-1)!\}$, where s is a fixed element of $[t_1, t_n]$. So in our notation we have $y_k(t,s) = \frac{(t-s)^k}{k!}$, and for each $k = 0, \ldots, n-1$, y_k is a solution to the initial value problem $Ly = 0$, $y_k^{(j)} = \delta_{jk}$, $0 \le j \le n-1$.

This will simplify our Green's function considerably, since $y_k^{(j)} = y_{k-j}$, for

$j \leq k$ and $y_k^{(j)} = 0$ for $j > k$. Also we have that

$$D = \begin{vmatrix} y_1'(t_2, t_1) & y_2'(t_2, t_1) & \dots & y_{n-1}'(t_2, t_1) \\ y_1''(t_3, t_1) & y_2''(t_3, t_1) & \dots & y_{n-1}''(t_3, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_n, t_1) & y_2^{(n-1)}(t_n, t_1) & \dots & y_{n-1}^{(n-1)}(t_n, t_1) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & y_1(t_2, t_1) & \dots & y_{n-2}(t_2, t_1) \\ 0 & 1 & \dots & y_{n-3}(t_3, t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix}$$

So $D = 1$.

Example $n = 3$:

When $n = 3$ our differential equation $(-1)^{n-1}Ly = 0$ becomes $Ly = y^{(3)} = 0$ with boundary conditions $y(t_1) = y'(t_2) = y''(t_3) = 0$. We will show that hypothesis (H) holds under the condition that $(t_2 - t_1) \geq (t_3 - t_2)$. It can be shown that if $(t_3 - t_2) > (t_2 - t_1)$ then hypothesis (h) does not hold.

From Theorem 5 we have that for this equation, our Green's function is

$$G(t,s) = \begin{cases} \text{for } s \in [t_1, t_2] & \\ \begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t_1 \leq t \leq s \leq t_2, \\ \begin{vmatrix} y_2(t,s) & y_1(t,t_1) & y_2(t,t_1) \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t_1 \leq s \leq t \leq t_3, \\ \text{for } s \in [t_2, t_3] & \\ \begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t_1 \leq t \leq s \leq t_3, \\ \begin{vmatrix} y_2(t,s) & y_1(t,t_1) & y_2(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t_2 \leq s \leq t \leq t_3. \end{cases}$$

For hypothesis (H) we need to show that $G(t,s) > 0$ for $t \in (t_1, t_3]$, $s \in (t_1, t_3)$, and that $G'(t_1, s) > 0$ for $s \in (t_1, t_3)$. We will first show that $G'(t_1, s) > 0$ for $s \in (t_1, t_3)$. First, let $s \in (t_1, t_2]$. Then we have

$$G'(t_1,s) = \begin{vmatrix} 0 & y_1'(t_1,t_1) & y_2'(t_1,t_1) \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} y_1(t_2,s) & y_1(t_2,t_1) \\ 1 & 1 \end{vmatrix} = y_1(t_2,t_1) - y_1(t_2,s)$$

$$= (t_2 - t_1) - (t_2 - s) = s - t_1 > 0.$$

If we have that $s \in [t_2, t_3)$, then

$$G'(t_1,s) = \begin{vmatrix} 0 & y_1'(t_1,t_1) & y_2'(t_1,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & y_1(t_2,t_1) \end{vmatrix} = y_1(t_2,t_1) = (t_2 - t_1) > 0.$$

Thus for $s \in (t_1, t_3)$ we have that $G'(t_1, s) > 0$.

We will now show why the condition $(t_3 - t_2) \leq (t_2 - t_1)$ will insure us that $G(t,s) > 0$ when $t \in (t_1, t_3]$, $s \in (t_1, t_3)$. We have two cases to consider, when $s \in (t_1, t_2]$ and $s \in [t_2, t_3)$.

Case 1) Fix $s \in (t_1, t_2]$.

If $t \in (t_1, s]$ then we have $G(t_1, s) = 0$, $G'(t_1, s) > 0$ and

$$G''(t,s) = \begin{vmatrix} 0 & 0 & 1 \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = -1.$$

So $G(t,s)$ is concave down on $(t_1, s]$ and $G(t_1, s) = 0$ and $G'(t_1, s) > 0$. Thus, if $G(s,s) > 0$, then $G(t,s) > 0$ for all $t \in (t_1, s]$. Now $G(t,s)$ is continuous at $t = s$, so if we can show that $G(t,s) > 0$ for all $t \in [s, t_3]$ then we will have that $G(t,s) > 0$ for all $t \in (t_1, t_3]$, where s is fixed in $(t_1, t_2]$.

Let $t \in [s, t_3]$ and define

$$f(t) = \begin{vmatrix} y_2(t,s) & y_1(t,t_1) & y_2(t,t_1) \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix},$$

for $t \in [t_1, t_3]$. Now $f(t)$ is a three times differentiable function and $f(t) \equiv G(t,s)$ for $t \in [s, t_3]$. Thus, $f(t)$ is a solution to our differential equation $Ly = 0$ and

satisfies the boundary conditions $y'(t_2) = 0$ and $y''(t_3) = 0$. Since $f'''(t) = 0$, $f''(t)$ is equal to a constant. But $f''(t_3) = 0$ so $f''(t) \equiv 0$ and we have that $f'(t)$ is equal to a constant. But $f'(t_2) = 0$ so $f'(t) \equiv 0$. Thus $f(t)$ is equal to a constant on $[t_1, t_3]$. Evaluating $f(t)$ at t_1 gives us

$$f(t_1) = \begin{vmatrix} y_2(t_1,s) & y_1(t_1,t_1) & y_2(t_1,t_1) \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} y_2(t_1,s) & 0 & 0 \\ y_1(t_2,s) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = y_2(t_1,s).$$

So $f(t) = f(t_1) = y_2(t_1,s) = \frac{(t_1-s)^2}{2!} > 0$, since $s \in (t_1, t_2]$. Thus $G(t,s) > 0$ for $t \in [s, t_3]$ when $s \in (t_1, t_2]$. So we have that when $s \in (t_1, t_2]$, $G(t,s) > 0$ for all $t \in (t_1, t_3]$.

Case 2) Fix $s \in [t_2, t_3)$.

When $t \leq s$, we have that $G(t_1,s) = 0$, $G'(t_1,s) > 0$, and like in case 1, $G''(t,s) = -1$. So, like before, we only need to consider $G(t,s)$ when $t \in [s, t_3]$.

Let $t \in [s, t_3]$ and define

$$f(t) = \begin{vmatrix} y_2(t,s) & y_1(t,t_1) & y_2(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix},$$

for $t \in [t_1, t_3]$. So $f(t) = G(t,s)$ when $t \in [s, t_3]$. Again we know that $f'''(t) = 0$ and that $f''(t_3) = 0$. Thus $f'(t)$ is a constant. Evaluating $f'(t)$ at s we have

$$f'(s) = \begin{vmatrix} y_1(s,s) & 1 & y_1(s,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & y_1(s,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= y_1(t_2,t_1) - y_1(s,t_1) = (t_2 - t_1) - (s - t_1) = t_2 - s \leq 0.$$

Thus $f'(t) \leq 0$ so $f(t)$ is non-increasing on $[t_1, t_3]$. So if $f(t_3) > 0$ then we would have what we want, $0 < f(t) = G(t,s)$ for $t \in [s, t_3]$. If we expand $f(t_3)$ along the

first column, we get

$$
\begin{aligned}
f(t_3) &= \begin{vmatrix} y_2(t_3,s) & y_1(t_3,t_1) & y_2(t_3,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = y_2(t_3,s) + \begin{vmatrix} y_1(t_3,t_1) & y_2(t_3,t_1) \\ 1 & y_1(t_2,t_1) \end{vmatrix} \\
&= y_2(t_3,s) + \{y_1(t_3,t_1)y_1(t_2,t_1) - y_2(t_3,t_1)\} \\
&= \frac{(t_3-s)^2}{2!} + \left\{(t_3-t_1)(t_2-t_1) - \frac{(t_3-t_1)^2}{2!}\right\} \\
&= \frac{(t_3-s)^2}{2!} + \frac{(t_3-t_1)}{2!}\{(t_2-t_1)-(t_3-t_2)\}.
\end{aligned}
$$

Now $\frac{(t_3-s)^2}{2!}$ is positive, but may be small, so to insure that $f(t_3) > 0$ we must have $(t_3 - t_2) \le (t_2 - t_1)$. Thus if this holds, we have that $f(t) > 0$ and so $G(t,s) > 0$ for all $t \in (t_1, t_3]$ and $s \in [t_2, t_3)$.

Hence for the boundary value problem, $Ly = y^{(3)} = 0$ and $Ty = 0$, we have that hypothesis (H) holds provided that $(t_3 - t_2) \le (t_2 - t_1)$.

Example $n = 4$:

In our final example we will take our differential equation to be $(-1)^{n-1}Ly = -y^{(4)} = 0$, with boundary conditions $y^{(i-1)}(t_i) = 0$, for $i = 1,2,3$ and 4. Under the conditions $(t_2 - t_1) \ge (t_4 - t_2)$ and $(t_3 - t_2) \ge (t_4 - t_3)$, we have that hypothesis (H) holds. For this equation we have that the Green's function is

$$G(t,s) = \begin{cases} \text{for } s \in [t_1, t_2] & \\ -\begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t_1 \le t \le s \le t_2 \\ -\begin{vmatrix} y_3(t,s) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t_1 \le s \le t \le t_4 \\ \text{for } s \in [t_2, t_3] & \\ -\begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t_1 \le t \le s \le t_3 \\ -\begin{vmatrix} y_3(t,s) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t_2 \le s \le t \le t_4 \\ \text{for } s \in [t_3, t_4] & \\ -\begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t_1 \le t \le s \le t_4 \\ -\begin{vmatrix} y_3(t,s) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t_3 \le s \le t \le t_4 \end{cases}$$

We will first show that $G'(t_1,s) > 0$ for all $s \in (t_1, t_4)$. In all cases, we have that the first row of $G'(t,s)$,

$\vec{\mathbf{R}}_1'(t) = (0, y_1'(t,t_1), y_2'(t,t_1), y_3'(t,t_1)) = (0, 1, y_1(t,t_1), y_2(t,t_1))$. So $\vec{\mathbf{R}}_1'(t_1) =$

$(0,1,0,0)$. If we expand $G'(t_1,s)$ along the first row, we have

$$G'(t_1,s) = \begin{cases} \begin{vmatrix} y_2(t_2,s) & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t_1 \le s \le t_2, \\ \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t_2 \le s \le t_3, \\ \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t_3 \le s \le t_4. \end{cases}$$

If we consider $G'(t_1,s)$ as a function of s, we can define functions $h_i(s)$ on $[t_1,t_4]$ for $i=1,2,3$ to be

$$h_1(s) = \begin{vmatrix} y_2(t_2,s) & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} \quad t_1 \le s \le t_4,$$

$$h_2(s) = \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} \quad t_1 \le s \le t_4,$$

$$h_3(s) = \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} \quad t_1 \le s \le t_4.$$

Then $h_i(s) = G'(t_1,s)$ when $s \in [t_i,t_{i+1}]$, for $i=1,2,3$. We will need to take the derivative of these functions so we note that $(\frac{d}{ds})^{(j)} y_k(t,s) = (-1)^j y_{k-j}(t,s)$ if $k \ge j$ and zero otherwise. Also we have that $h_1(t_2) = h_2(t_2)$ and $h_2(t_3) = h_3(s)$, since h_3 is a constant function.

Now $h_1(t_1) = 0$ since, in this determinant, the first and last columns are

equal. Also, we have that

$$h_1'(s) = -\begin{vmatrix} y_1(t_2,s) & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 1 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 1 \end{vmatrix},$$

so that $h_1'(t_1) = 0$ since, in this case, the first and second columns are equal.

Finally,

$$h''(s) = \begin{vmatrix} 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

The last equation gives us that h_1' is increasing on $[t_1,t_4]$. Now $h_1'(t_1) = 0$ so $h_1' > 0$ on $(t_1,t_4]$. So h_1 in increasing on this interval and $h_1(t_1) = 0$. Thus we have shown, in particular, that $h_1(s) > 0$ for all s in $(t_1,t_2]$.

Now

$$h_2'(s) = -\begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 1 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 1 \end{vmatrix} = y_1(t_2,t_1) = (t_2 - t_1) > 0.$$

So h_2 is an increasing function with $h_2(t_2) = h_1(t_2) > 0$. Thus h_2 is positive on $[t_2,t_3]$.

Finally, h_3 is constant and $h_3(s) = h_2(t_3) > 0$. So h_3 is positive on $[t_3,t_4]$. Putting this all together we have that $G'(t_1,s) > 0$ for all s in (t_1,t_4).

We will now show why the conditions $(t_2 - t_1) \geq (t_4 - t_2)$ and $(t_3 - t_2) \geq (t_4 - t_3)$ insure us that $G(t,s) > 0$ for $(t,s) \in (t_1,t_4] \times (t_1,t_4)$. We have three cases to consider.

Case 1: Fix $s \in (t_1,t_2]$

For $t_1 < t \le s$ we have that

$$G''(t,s) = -\begin{vmatrix} 0 & 0 & 1 & y_1(t,t_1) \\ y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 1 & y_1(t,t_1) \\ y_1(t_3,s) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= y_1(t_3,s) - \{y_1(t_3,t_1) - y_1(t,t_1)\}$$

$$= (t_3 - s) - (t_3 - t_1) + (t - t_1)$$

$$= t - s \le 0 \quad \text{since } t \le s.$$

So $G(t,s)$ is concave down on $(t_1,s]$, $G(t_1,s) = 0$ and we have already shown that $G'(t_1,s) > 0$. Thus if $G(s,s) > 0$, then $G(t,s) > 0$ for all t in $(t_1,s]$. Now, since $G(t,s)$ is continuous at $t = s$, we only need to show that $G(t,s) > 0$ for $t \ge s$.

We know that in $t, t \ne s$, $G(t,s)$ satisfies $y^{(4)} = 0$. Also, since $t \ge s$, $s \in (t_1,t_2]$, $G(t,s)$ will satisfy the boundary conditions $y^{(i-1)}(t_i) = 0$, for $i = 2,3$ and 4. So $G^{(4)}(t,s) = 0$ which tells us that $G^{(3)}(t,s)$ is a constant. But $G^{(3)}(t_4,s) = 0$ so $G^{(3)}(t,s) \equiv 0$ which tells us that $G''(t,s)$ is a constant. But again, $G''(t_3,s) = 0$ so $G'(t,s)$ is a constant. Finally, $G'(t_2,s) = 0$ so we have that $G(t,s)$ is constant for $t \ge s$. Now, let $f(t)$ be

$$f(t) = -\begin{vmatrix} y_3(t,s) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}.$$

So we have that $f(t) = G(t,s)$ when $t \in [s, t_4]$. Evaluating f at t_1 gives us

$$f(t_1) = -\begin{vmatrix} y_3(t_1,s) & y_1(t_1,t_1) & y_2(t_1,t_1) & y_3(t_1,t_1) \\ y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} y_3(t_1,s) & 0 & 0 & 0 \\ y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -y_3(t_1,s) = -\frac{(t_1-s)^3}{3!} > 0, \text{ since } t_1 < s.$$

Thus $f(t) > 0$ for all t in $[t_1, t_4]$. So $G(t,s) > 0$ when $t \in [s, t_4]$. But this implies that $G(t,s) > 0$ for all $t \in (t_1, t_4]$ when $s \in (t_1, t_2]$.

Case 2: Fix $s \in [t_2, t_3]$. Let $t \in (t_1, s]$ and consider $G''(t,s)$. As in Case 1, we will have that $G''(t,s) = t - s \leq 0$. This can be easily seen since the only difference between this Green's function and the one in Case 1, is the element $y_2(t_2,s)$, which lies in the second row, first column slot. After taking two derivatives of $G(t,s)$, we will expand along the second column, which has only one nonzero element, in the second slot. This will eliminate the element $y_2(t_2,s)$, and $G''(t,s)$ will be the same as in Case 1. Thus $G''(t,s) \leq 0$, so $G(t,s)$ is concave down on $(t_1, s]$. Since $G(t_1,s) = 0$ and $G'(t_1,s) > 0$, we only have to show that $G(s,s) > 0$. But then, by continuity, we only need to show that $G(t,s) > 0$ for $t \in [s, t_4]$.

We now let $t \in [s, t_4]$. We know that $G(t,s)$ is a solution to $y^{(4)} = 0$ on $(s, t_4]$ and satisfies the appropriate boundary conditions. So we have that $G^{(4)}(t,s) \equiv 0$

and $G^{(3)}(t_4,s) = G''(t_3,s) = 0$. This gives us that $G'(t,s)$ is a constant function. Let $f(t)$ be defined on $t \in [t_1, t_4]$ by

$$f(t) = -\begin{vmatrix} y_3(t,s) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

So $f(t) = G(t,s)$ when $t \geq s$. This then gives us that $f'(t)$ is a constant function. Evaluating f' at t_2 and using propert. s of determinants we have

$$f'(t_2) = -\begin{vmatrix} y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} y_2(t_2,s) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$- \begin{vmatrix} 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -y_2(t_2,s) = -\frac{(t_2-s)^2}{2!} \leq 0.$$

So $f'(t) = G'(t,s) \leq 0$. Thus $G(t,s)$ is a nonincreasing function on $(s, t_4]$. So if $G(t_4,s) > 0$, then we would have that $G(t,s) > 0$ for all t in $(t_1, t_4]$.

At this point we ask ourself, which s value in $[t_2, t_3]$ give us the 'least positive' $G(t_4,s)$ value? Considering $\frac{d}{ds}G(t_4,s)$ as a function in s we have

$$\frac{d}{ds}G(t_4,s) = -\begin{vmatrix} -y_2(t_4,s) & y_1(t_4,t_1) & y_2(t_4,t_1) & y_3(t_4,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ -1 & 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} y_2(t_4,s) & y_1(t_4,t_1) & y_2(t_4,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= y_2(t_4,s) + \{y_1(t_4,t_1)y_1(t_2,t_1) - y_2(t_4,t_1)\}$$

$$= \frac{(t_4-s)^2}{2!} + \left\{(t_4-t_1)(t_2-t_1) - \frac{(t_4-t_1)^2}{2!}\right\}$$

$$= \frac{(t_4-s)^2}{2!} + \frac{(t_4-t_1)}{2!}\{(t_2-t_1)-(t_4-t_2)\}.$$

Since we required that $(t_2 - t_1) \geq (t_4 - t_2)$ then we have that $\frac{d}{ds}G(t_4, s) > 0$ for all $s \in [t_2, t_3]$. Thus under this requirement we have that $G(t_4, s)$ is an increasing function in s, so $G(t_4, t_2^+) \leq G(t_4, s)$ for all $s \in [t_2, t_3]$. Now we know that $G(t, s)$ is a continuous function in both t and s and so $G(t_4, t_2^+) = G(t_4, t_2^-) = G(t_4, t_2)$. And we proved in the previous case that $G(t, s) > 0$ for all $t \in (t_1, t_4]$, $s \in (t_1, t_2]$. Thus $0 < G(t_4, t_2) \leq G(t_4, s)$ for all $s \in [t_2, t_4]$ provided that $(t_2 - t_1) \geq (t_4 - t_2)$.

Summing up, we have shown that if $(t_2 - t_1) \geq (t_4 - t_2)$, then $G(t, s) > 0$ for all t in $(t_1, t_4]$, s fixed in $[t_2, t_3]$.

Our final case is when s is an element of $[t_3, t_4)$.

<u>Case 3</u>: Fix $s \in [t_3, t_4)$.

Let $t \leq s$ and consider $G''(t,s)$ which is

$$G''(t,s) = -\begin{vmatrix} 0 & 0 & 1 & y_1(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & y_1(t,t_1) \\ 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \end{vmatrix} = -\begin{vmatrix} 1 & y_1(t,t_1) \\ 1 & y_1(t_3,t_1) \end{vmatrix}$$

$$= y_1(t,t_1) - y_1(t_3,t_1) = (t-t_1) - (t_3-t_1)$$

$$= (t-t_3).$$

This gives that $G(t,s)$ is concave down on $[t_1,t_3)$ and concave up on $(t_3,s]$. Since $G(t_1,s)=0$ and $G'(t_1,s)>0$ then all we have to worry about is the sign of $G(t,s)$ for $t \in [t_3,s]$. We know $G'(t_2,s)=0$ and $G''(t,s) = t - t_3$, so $G'(t,s) < 0$ on $(t_2,t_3]$ and then begins to increase. Now, if $g(t)$ is a third order polynomial, it is easy to see that if $g'(a)=0$ and $g''(b)=0$, then $g'(c)=0$, where $c = b + (b-a)$. Thus, since $G(t,s)$ is a third order polynomial and $G'(t_2,s)=0$ and $G''(t_3,s)=0$ we must have that $G'(t_\circ,s)=0$ where $t_\circ = t_3 + (t_3 - t_2)$. {This is easily verified for $G(t,s)$, although algebraically horrendous.}

Now we know that $s \leq t_3 + (t_3 - t_2)$ since we required that $t_4 \leq t_3 + (t_3 - t_2)$, that is $(t_4 - t_3) \leq (t_3 - t_2)$. Thus we have that $G(t,s)$ will be decreasing on $(t_2,s]$. This gives us that if $G(s,s)>0$, then $G(t,s)>0$ for all $t \in (t_1,s]$. Again by continuity of $G(t,s)$ at $t=s$, we only need to show that $G(t,s)>0$ for all $t \in [s,t_4]$.

Let t be in the interval $[s, t_4]$ and consider $G''(t,s)$ which is

$$G''(t,s) = -\begin{vmatrix} y_1(t,s) & 0 & 1 & y_1(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} y_1(t,s) & 1 & y_1(t,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= -y_1(t,s) - \{y_1(t_3,t_1) - y_1(t,t_1)\}$$

$$= -(t-s) - \{(t_3 - t_1) - (t - t_1)\}$$

$$= (s - t_3) \geq 0, \quad \text{since } t_3 \leq s < t_4.$$

This gives us that $G'(t,s)$ is nondecreasing for $t \in [s, t_4]$. If we could show that $G'(t_4, s) \leq 0$ then $G(t,s)$ would be a decreasing function on $[s, t_4]$. Then, if $G(t_4, s) > 0$ we would have that $G(t,s) > 0$ for $t \in [s, t_4]$. Consider

$$G'(t_4,s) = -\begin{vmatrix} y_2(t_4,s) & 1 & y_1(t_4,t_1) & y_2(t_4,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -y_2(t_4,s) + \begin{vmatrix} 1 & y_1(t_4,t_1) & y_2(t_4,t_1) \\ 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \end{vmatrix}.$$

Now $-y_2(t_4, s) = \{(t_4 - s)^2/2!\} \leq 0$ so we will only consider the determinant term. We now define the function $h(r)$, to be the determinant term with t_1 replaced by

r. So

$$h(r) = \begin{vmatrix} 1 & y_1(t_4, r) & y_2(t_4, r) \\ 1 & y_1(t_2, r) & y_2(t_2, r) \\ 0 & 1 & y_1(t_3, r) \end{vmatrix}, \quad \text{which gives}$$

$$\begin{aligned} h'(r) &= - \begin{vmatrix} 1 & 1 & y_2(t_4, r) \\ 1 & 1 & y_2(t_2, r) \\ 0 & 0 & y_1(t_3, r) \end{vmatrix} - \begin{vmatrix} 1 & y_1(t_4, r) & y_1(t_4, r) \\ 1 & y_1(t_2, r) & y_1(t_2, r) \\ 0 & 1 & 1 \end{vmatrix} \\ &= 0. \end{aligned}$$

Thus $h(r)$ is a constant. Evaluating h at t_4 gives us

$$\begin{aligned} h(t_4) &= \begin{vmatrix} 1 & y_1(t_4, t_4) & y_2(t_4, t_4) \\ 1 & y_1(t_2, t_4) & y_2(t_2, t_4) \\ 0 & 1 & y_1(t_3, t_4) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & y_1(t_2, t_4) & y_2(t_2, t_4) \\ 0 & 1 & y_1(t_3, t_4) \end{vmatrix} \\ &= y_1(t_2, t_4) y_1(t_3, t_4) - y_2(t_2, t_4) \\ &= (t_2 - t_4)(t_3 - t_4) - \frac{(t_2 - t_4)^2}{2!} \\ &= (t_4 - t_2)(t_4 - t_3) - \frac{(t_4 - t_2)^2}{2!} \\ &= \frac{(t_4 - t_2)}{2!} \left\{ (t_4 - t_3) - (t_3 - t_2) \right\}. \end{aligned}$$

Thus we have that $h(r) \leq 0$,{and so our determinant is ≤ 0}, provided that $(t_4 - t_3) \leq (t_3 - t_2)$, our earlier constraint! This gives us that $G'(t_4, s) \leq 0$, so we have that $G(t, s)$ is decreasing in t on $[s, t_4]$ provided that $(t_4 - t_3) \leq (t_3 - t_2)$. Hence if $G(t_4, s) > 0$, then $G(t, s) > 0$ for all t in $[s, t_4]$.

If we consider $G(t_4, s)$ as a function of s, then we wish to find the s value which will give us the 'least positive' value of $G(t_4, s)$. Taking the derivative with respect to s gives us

$$\frac{d}{ds}G(t_4,s) = -\begin{vmatrix} -y_2(t_4,s) & y_1(t_4,t_1) & y_2(t_4,t_1) & y_3(t_4,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= y_2(t_4,s) = \frac{(t_4-s)^2}{2!} > 0.$$

Thus $G(t_4,s)$ is increasing in s for s in $[t_3,t_4)$. This gives us that $G(t_4,t_3) \leq G(t_4,s)$ for all $s \in [t_3,t_4)$. But from Case 2) and continuity we know that $G(t_4,t_3) > 0$ provided that $(t_2-t_1) \geq (t_4-t_2)$. Hence we have that if $(t_2-t_1) \geq (t_4-t_2)$ and $(t_3-t_2) \geq (t_4-t_3)$ then $G(t,s) > 0$ for all $t \in (t_1,t_4]$, $s \in [t_3,t_4)$.

Thus, combining all of our cases, we have shown that if we have $(t_2-t_1) \geq (t_4-t_2)$ and $(t_3-t_2) \geq (t_4-t_3)$ then $G(t,s) > 0$ for all $t \in (t_1,t_4]$, $s \in (t_1,t_4)$. Since we also showed that $G'(t_1,s) > 0$ for all $s \in (t_1,t_4)$, we have that when $Ly = y^{(4)}$ we have that hypothesis (H) holds.

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Chapter 4

Applications to Difference Equations

I) INTRODUCTION

In this chapter we will show how the results from our last chapter will also hold for an n-th order linear difference equation. Many of the definitions and notation used will be from Hartman [9], and Hankerson and Peterson [6,7]. In general, interval notation will specify an interval of integers. So, for example, $[a, b)$ will mean the set of integers $\{a, a+1, a+2, \ldots, b-2, b-1\}$.

Let n be an integer greater than or equal to two and k a fixed integer with $1 \leq k \leq n-1$. We define the n-th order linear difference equation

$$(1) \qquad Ly(t) \doteq \sum_{i=0}^{n} \alpha_i(t) y(t-k+i) = 0, \quad t \in [a+k, b+k]$$

where we assume the coefficients $\alpha_i(t)$ are defined on $[a+k, b+k]$, for $i = 1, 2, \ldots, n$, $\alpha_n(t) \equiv 1$, and $\alpha_o(t)$ satisfies

$$(2) \qquad (-1)^n \alpha_o(t) > 0,$$

for $t \in [a+k.b+k]$. We note that solutions to our difference equation (1) are defined on $[a, b+n]$.

Condition (2) implies $\alpha_o(t) \neq 0$ for all $t \in [a+k, b+k]$, and this guarantees that solutions to the intital value problem

$$Ly(t) = h(t)$$

$$y(t_o + i) = \dot{y_i}, 0 \leq i \leq n-1,$$

for $t_o \in [a, b]$, exist on $[a, b+n]$, and that (1) has exactly n independent solutions on $[a, b+n]$.

We define the difference operator Δ, by $\Delta y(t) = y(t+1) - y(t)$. We can then recursively define the operators $\Delta^i y(t) = \Delta(\Delta^{i-1} y(t))$ for $i = 1, 2, \ldots,$ where it is understood that $\Delta^0 y(t) = y(t)$. We note that by induction, we can also define the the *ith* order difference operator Δ^i, by

$$\Delta^i y(t) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} y(t+i-j).$$

Hartman [9] gives us the following definition.

<u>Definition</u>: Let $y(t)$ be a solution of (1). We say that y has a *generalized zero* at t_o if either $y(t_o) = 0$ or there exists an integer j, with $1 \leq j \leq t_o - a$ such that

$$(-1)^j y(t_o - j) y(t_o) > 0, \text{ and, if } j > 1,$$

$$y(t) = 0, \text{ for } t_o - j < t < t_o.$$

II) <u>THE GREEN'S FUNCTION</u>:

Let $m \geq 1$ and define the n-th order vector difference equation $Lu(t) \doteq \sum_{i=0}^{n} \alpha_i(t) u(t-k+i)$, $t \in [a+k, b+k]$ where $u(t)$ is an m-column vector such

that $u: [a, b+n] \to \mathcal{R}^m$ and the α_i's are as in (1). Also, let $P(t) = (p_{ij}(t))$, $Q(t) = (q_{ij}(t))$ be discrete $m \times m$ matrix functions on $[a+k, b+k]$ and let $a = t_1 < t_2 < \cdots < t_n = b+1$.

We consider the n-point right focal eigenvalue problem:

$$(3) \qquad (-1)^{n-1} Lu = \lambda P(t) u$$

$$Tu = 0,$$

where $Tu = 0$ denotes the boundary conditions $\Delta^{i-1} y(t_i) = 0$, $i = 1, 2, \ldots, n$, and $a = t_1 < t_2 < \cdots < t_n = b+1$. The Green's function for the scalar difference boundary value problem

$$(4) \qquad (-1)^{n-1} Ly = 0$$

$$Ty = 0$$

where Ly and Ty are as above, but defined appropriately for the scalar case, has different properties then its differential equation analog. These properties, given in Hartman [9], are in the following lemma.

LEMMA 1. *Suppose the function $G(t,s)$ has the properties:*

i) $G(t,s)$ is defined on $[a,b] \times [a+k, b+k]$;

ii) For each fixed $s \in [a+k, b+k]$, $LG(t,s) = (-1)^{n-1}\delta_{ts}$ for all $t \in [a, b+n]$, where δ_{ts} is the Kronecker-delta function;

iii) For each fixed $s \in [a+k, b+k]$, $\Delta^{i-1} G(t_i, s) = 0$, $i = 1, 2, \ldots, n$.

Then, for $h(t)$ defined on $[a+k, b+k]$, we have that $y(t) = \sum_{s=a+k}^{b+k} G(t,s)h(s)$ solves

$$(-1)^{n-1} Ly(t) = h(t)$$

$$Ty = 0.$$

PROOF: The proof is straight forward. For $t \in [a+k, b+k]$,

$$\begin{aligned}
(-1)^{n-1} Ly(t) &= (-1)^{n-1} L\Big(\sum_{s=a+k}^{b+k} G(t,s)h(s)\Big) \\
&= (-1)^{n-1} \sum_{s=a+k}^{b+k} LG(t,s)h(s) \\
&= (-1)^{n-1} \sum_{s=a+k}^{b+k} (-1)^{n-1} \delta_{ts} h(s) \\
&= h(t)
\end{aligned}$$

The boundary conditions are satisfied by condition iii) in our definition.

Similiar to differential equations, we now define what it means for a difference equation to be right disfocal.

Definition: The difference equation $Ly = 0$ is said to be *right disfocal* on an interval $[a, b+n]$, if there does not exist a nontrivial solution y of $Ly = 0$ and points $t_1 \le t_2 \le \cdots \le t_n \in [a, b+1]$, such that $\Delta^{i-1} y$ has a generalized zero at t_i, $1 \le i \le n$.

We now introduce some more notation. For each fixed integer s in the interval $[a, b+1]$, let $\{y_0(t,s), y_1(t,s), \ldots, y_{n-1}(t,s)\}$ be the set of (linearly independent)

solutions of $Ly = 0$, where $\Delta^i y_j(t,s)|_{t=s} = \delta_{jk}$, $0 \le j,k \le n-1$. This tells us that $y_j(t,s)$ has j zeros, at $s, s+1, \dots, s+j-1$ and $y_j(s+j,s) = 1$.

One more final bit of notation. For $j = 1, 2, \dots, n-1$, define the interval I_j of integers by

$$I_j = \begin{cases} [t_1 + k, t_2 + k - 1], & \text{for } j = 1 \\ [t_j + k - 1, t_{j+1} + k - 1], & \text{for } 2 \le j \le n-1. \end{cases}$$

Let $Ly = 0$ be right disfocal. Then, for each fixed $s \in I_j$, $t \in [a, b+n]$, we define the functions $u_j(t)$, $v_j(t)$, for $j = 1, 2, \dots, n-1$, by

$$u_j(t) = \frac{(-1)^{n-1}}{D} \begin{vmatrix} 0 & y_1(t,t_1) & \dots & y_{n-1}(t,t_1) \\ 0 & \Delta y_1(t_2,t_1) & \dots & \Delta y_{n-1}(t_2,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \Delta^{j-1} y_1(t_j,t_1) & \dots & \Delta^{j-1} y_{n-1}(t_j,t_1) \\ \Delta^j y_{n-1}(t_{j+1},\hat{s}) & \Delta^j y_1(t_{j+1},t_1) & \dots & \Delta^j y_{n-1}(t_{j+1},t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{n-1} y_{n-1}(t_n,\hat{s}) & \Delta^{n-1} y_1(t_n,t_1) & \dots & \Delta^{n-1} y_{n-1}(t_n,t_1) \end{vmatrix}$$

where $\hat{s} = s - k + 1$ and $v_j(t)$ is the same as $u_j(t)$ except we replace the zero in the first row, first column by $y_{n-1}(t, s-k+1)$. In the above formula, D is given by

$$D = \begin{vmatrix} \Delta y_1(t_2,t_1) & \Delta y_2(t_2,t_1) & \dots & \Delta y_{n-1}(t_2,t_1) \\ \Delta^2 y_1(t_3,t_1) & \Delta^2 y_2(t_3,t_1) & \dots & \Delta^2 y_{n-1}(t_3,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{n-1} y_1(t_n,t_1) & \Delta^{n-1} y_2(t_n,t_1) & \dots & \Delta^{n-1} y_{n-1}(t_n,t_1) \end{vmatrix}.$$

The functions u_j, v_j, $j = 1, 2, \dots, n-1$, are well defined provided that $D \ne 0$. As in the last chapter, $Ly = 0$ being right disfocal guarantees us that $D \ne 0$. To see this, we again suppose that $D = 0$, and let $\mathbf{A} = (\Delta^i y_j(t_{i+1}, t_1))$, for

$1 \le i, j \le n-1$. Then we have that $|\mathbf{A}| = D$, where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$. Since $D = 0$, that is $|\mathbf{A}| = 0$, we know that there exists a nontrivial column vector $\vec{\mathbf{C}} = (C_1, C_2, \ldots, C_{n-1})^T$ so that $\mathbf{A}\vec{\mathbf{C}} = \vec{0}$. Let $z(t) = C_1 y_1(t, t_1) + C_2 y_2(t, t_1) + \cdots + C_{n-1} y_{n-1}(t, t_1)$. Since $z(t)$ is a linear combination of solutions of $Ly = 0$, we have by linearity of L, that $Lz = 0$. Now $z(t_1) = 0$ since $y_j(t_1, t_1) = 0$ for each $j = 1, 2, \ldots, n-1$. Also, $\Delta^i z(t_{i+1}) = 0$ for $i = 1, 2, \ldots, n-1$, since $\Delta^i z(t_{i+1})$ is the *i-th* row of $\mathbf{A}$ times the column vector $\vec{\mathbf{C}}$, and $\mathbf{A}\vec{\mathbf{C}} = \vec{0}$. Thus, $Lz = 0$ and $Tz = 0$ and z is not identically zero since $\vec{\mathbf{C}}$ is nontrivial. This contradicts $Ly = 0$ is right disfocal. Hence $D \neq 0$ and our functions u_j, v_j, for $j = 1, 2, \ldots, n-1$ are well defined. Now that we have established that $D \neq 0$, a standard argument shows that $D > 0$.

We note that since L is linear, u_j, v_j are, for each fixed s, solutions of $Ly = 0$. With our functions u_j, v_j defined, we can now go on to define our function $G(t, s)$.

LEMMA 2. *Assume that $Ly = 0$ is right disfocal on $[a, b+n]$. For each fixed $s \in I_j$, let*

$$G(t,s) = \begin{cases} u_j(t), & \text{for } t < s - k + n \\ v_j(t), & \text{for } t \ge s - k + n. \end{cases} \tag{5}$$

Then $G(t, s)$ satisfies the properties i)–iii) of Lemma 1.

PROOF: We need to show that $G(t, s)$ satisfies:

i) $G(t, s)$ is defined on $[a, b+n] \times [a+k, b+k]$.

ii) For each fixed $s \in [a+k, b+k]$, $LG(t, s) = (-1)^{n-1}\delta_{ts}$ for $t \in [a, b+n]$.

iii) For each fixed $s \in [a+k, b+k]$, $\Delta^{i-1}G(t_i, s) = 0$, for $i = 1, 2, \ldots, n$.

To show that $G(t,s)$ satisfies these properties, we first note that from the definitions of $u_j(t)$ and $v_j(t)$, we have that $v_j(t) - u_j(t) = (-1)^{n-1} y_{n-1}(t, s-k+1)$. Then since $y_{n-1}(s-k+1+i, s-k+1) = 0$ for $i = 0, 1, \ldots, n-2$, we have that $v_j(t) = u_j(t)$ for $t \in [s-k+1, s-k+n-1]$. Thus we can similarly define $G(t,s)$ as

$$G(t,s) = \begin{cases} u_j(t), & \text{for } t < s-k+n \\ v_j(t), & \text{for } t \geq s-k+1. \end{cases} \tag{6}$$

It is clear from our definition of $u_j(t)$ and $v_j(t)$ that $G(t,s)$ satisfies i). To show ii), let s be a fixed element of $[a+k, b+k]$, so $s \in I_j$ for some j. Let $t < s$. Then for $i = 0, 1, \ldots, n$, we have $t - k + i \leq t - k + n < s - k + n$, so we get from (1) and (6) that

$LG(t,s) = \sum_{i=0}^{n} \alpha_i(t) G(t-k+i, s) = \sum_{i=0}^{n} \alpha_i u_j(t-k+i) = Lu_j(t) = 0$, since u_j is a solution of $Ly = 0$.

If $t > s$, then $t \geq s+1$, so that for $i = 0, 1, \ldots, n$, we have $t - k + i \geq s + 1 - k + i \geq s - k + 1$, so again from (1) and (6) we have

$LG(t,s) = \sum_{i=0}^{n} \alpha_i(t) G(t-k+i, s) = \sum_{i=0}^{n} \alpha_i(t) v_j(t-k+i) = Lv_j(t) = 0$, since v_j is a solution of $Ly = 0$.

We now let $t = s$. Then since $s - k + i < s - k + n$ for $i = 0, 1, \ldots, n-1$, we have from (6) that $G(s-k+i, s) = u_j(s-k+i)$ for $i = 0, 1, \ldots, n-1$, and

$G(s-k+n,s) = v_j(s-k+n)$. Then

$$\begin{aligned}
LG(s,s) &= \sum_{i=0}^{n} \alpha_i(s)G(s-k+i,s) \\
&= \sum_{i=0}^{n-1} \alpha_i G(s-k+i,s) + \alpha_n(s)G(s-k+n,s) \\
&= \sum_{i=0}^{n-1} \alpha_i(s)u_j(s-k+i) + v_j(s-k+n), \quad \{\text{since } \alpha_n = 1\} \\
&= v_j(s-k+n) - u_j(s-k+n) + \sum_{i=0}^{n} \alpha_j(s)u_j(s-k+i) \\
&= (-1)^{n-1}y_{n-1}(s-k+n, s-k+1) + Lu_j(s) \\
&= (-1)^{n-1}y_{n-1}((s-k+1)+(n-1), s-k+1) \\
&= (-1)^{n-1}.
\end{aligned}$$

So $LG(s,s) = (-1)^{n-1}$.

Since s was an arbitrary element of $[a+k, b+k]$ we have that for each fixed $s \in [a+k, b+k]$, $LG(t,s) = (-1)^{n-1}\delta_{ts}$.

Lastly, we need to show that for each fixed $s \in [a+k, b+k]$, $\Delta^{i-1}G(t_i, s) = 0$ for $i = 1, 2, \ldots, n$. Fix $s \in [a+k, b+k]$, so $s \in I_j$ for some j. Consider $G(t_1, s)$. Now $t_1 + k \leq s$, so $t_1 \leq s-k < s-k+n$ which gives us from (6) that $G(t_1, s) = u_j(t_1)$. Since $y_j(t_1, t_1) = 0$ for $j = 1, 2, \ldots, n-1$, we have that the top row of the determinant which defines u_j, is all zeros and so $G(t_1, s) = 0$.

Now, consider $\Delta^{i-1}G(t_i, s)$ where $t_i \leq t_j$, $i \geq 2$, where this j is such that $s \in I_j$. For $\tau = 0, 1, \ldots, i-1$, we have that $t_i + i - 1 - \tau \leq t_i + (i-1) \leq$

$t_j + j - 1 \leq s - k + 1 + j - 1$ (since $t_j + k - 1 \leq s$) $\leq s - k + j < s - k + n$, since $j \leq n - 1$. So $t_i + i - 1 - \tau < s - k + n$, for $\tau = 0, 1, \ldots, i - 1$. Thus from (6), $G(t_i + i - 1 - \tau, s) = u_j(t_i - 1 - \tau)$. Hence, if we let $\hat{s} = s - k + 1$ then

$$\begin{aligned}
\Delta^{i-1}G(t_i,s) &= \Delta^{i-1}u_j(t_i) \\
&= \frac{(-1)^{n-1}}{D}\begin{vmatrix} 0 & \Delta^{i-1}y_1(t_i,t_1) & \ldots & \Delta^{i-1}y_{n-1}(t_i,t_1) \\ 0 & \Delta y_1(t_i,t_1) & \ldots & \Delta y_{n-1}(t_i,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{n-1}y_{n-1}(t_n,\hat{s}) & \Delta^{n-1}y_1(t_n,t_1) & \ddots & \Delta^{n-1}y_{n-1}(t_n,t_1) \end{vmatrix} \\
&= 0,
\end{aligned}$$

since the first and the i-*th* row are equal. Thus we have shown that $\Delta^{i-1}G(t_i, s) = 0$ for $i = 1, 2, \ldots, j$.

Now let $j < i \leq n$ so $t_{j+1} \leq t_i$. Then for $\tau = 0, 1, \ldots, i - 1$,

$$\begin{aligned}
t_i + i - 1 - \tau &\geq t_i \\
&\geq t_{j+1} \\
&\geq s - k + 1, \quad \text{since } s \leq t_{j+1} + k - 1.
\end{aligned}$$

Thus by (6) we have $G(t_i + i - 1 - \tau, s) = v_j(t_i + i - 1 - \tau)$. Hence, if we again let $\hat{s} = s - k + 1$, then

$$\begin{aligned}
\Delta^{i-1}G(t_i,s) &= \Delta^{i-1}v_j(t_i) \\
&= \frac{(-1)^{n-1}}{D}\begin{vmatrix} \Delta^{i-1}y_{n-1}(t_i,\hat{s}) & \Delta^{i-1}y_1(t_i,t_1) & \ldots & \Delta^{i-1}y_{n-1}(t_i,t_1) \\ 0 & \Delta y_1(t_i,t_1) & \ldots & \Delta y_{n-1}(t_i,t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{n-1}y_{n-1}(t_n,\hat{s}) & \Delta^{n-1}y_1(t_n,t_1) & \ddots & \Delta^{n-1}y_{n-1}(t_n,t_1) \end{vmatrix} \\
&= 0,
\end{aligned}$$

since the first and the i-*th* row are equal. Thus we have shown that $\Delta^{i-1}G(t_i,s) = 0$ for $j < i \leq n$. Combining the last two cases, we have shown that $\Delta^{i-1}G(t_i,s) = 0$ for $i = 1,2,\ldots,n$. Since s was a fixed but arbitrary element of $[a+k, b+k]$, we have shown that $G(t,s)$ satisfies condition iii) and this completes the proof of Lemma 2.

It is easy to see that if $Ly = 0$ is right disfocal, then the function $G(t,s)$ from Lemma 2 is unique. For suppose that $H(t,s)$ satisfies the properties i)–iii). Then for s a fixed but arbitrary element of $[t_1+k, t_n+k]$, define the function $w(t) = G(t,s) - H(t,s)$. From property ii), we have that $Lw(t) = L(G(t,s) - H(t,s)) = LG(t,s) - LH(t,s) = (-1)^{n-1}\delta_{t,s} - (-1)^{n-1}\delta_{t,s} = 0$. Also $\Delta^i w(t) = \Delta^i G(t,s) - \Delta^i H(t,s)$, so from property iii) we have that $\Delta^{i-1}w(t_i) = 0$ for $i = 1,2,\ldots,n$. Thus since $Ly = 0$ is right disfocal we must have that $w(t) = 0$ for all $t \in [a, b+n]$, and since s was an arbitrary element of $[a+k, b+k]$ we have that $G(t,s) = H(t,s)$ on $[a, b+n] \times [a+k, b+k]$. Hence $G(t,s)$ is unique.

We now define the Green's function for the boundary value problem (4).

Definition: If $Ly = 0$ is right disfocal, then the function satisfying the properties i)–iii) of Lemma 1, is called the *Green's function*, $G(t,s)$, for the boundary value problem (4).

This definition will allow us to summerize the last two lemmas in the following theorem.

THEOREM 3. *If $Ly = 0$ is right disfocal on $[a, b+n]$, then the boundary value*

problem

$$(-1)^{n-1}Ly = h(t)$$

$$\Delta^{i-1}y(t_i) = 0, \text{ for } i = 1, 2, \ldots, n$$

has a unique solution, $y(t)$, given by

$$y(t) = \sum_{s=a+k}^{b+k} G(t,s)h(s),\ t \in [a, b+n]$$

where $G(t,s)$ is the Green's function for $(-1)^{n-1}Ly = 0$, $\Delta^{i-1}y(t_i) = 0$, $1 \le i \le n$, and is given by (6).

We will close this section with the following hypothesis.

Hypothesis (H): Let the difference equation $Ly = 0$ be right disfocal on $[a, b+n]$. We will assume that the Green's function for (4) satisfies $G(t,s) > 0$ for $t \in (a, b+n]$, $s \in [a+k, b+k]$.

This hypothesis is not true in all cases, but we will show sufficient conditions for (H) to hold for $n = 2, 3$ and 4.

III)EXISTENCE AND COMPARISON THEOREMS

Our results for difference equations are similar to those for differential equations. We must first introduce a suitable Banach space for our difference equation, eigenvalue problem (3). Let $\mathcal{B} = \{u : [a+k, b+k] \to \mathcal{R}^m\}$ with norm $||u|| = \max_{[a+k,b+k]} |u(t)|$, where $|\cdot|$ is the Euclidean norm. Following the ideas from Hankerson and Peterson [6,7], and papers by Tomastik [17,18], we

let $I, J \subseteq \{1, 2, \ldots, m\}$ be such that $I \cup J = \{1, 2, \ldots, m\}$ and $I \cap J = \emptyset$. (It is permissible for $I = \emptyset$ or $J = \emptyset$.) Let $\mathcal{K}$ be the 'quadrant' cone in $\mathcal{R}^m$ defined by $\mathcal{K} = \{x = (x_1, x_2, \ldots, x_m) \,|\, x_i \geq 0 \text{ if } i \in I, x_i \leq 0 \text{ if } i \in J\}$.

Although some of our results will hold for any solid cone in $\mathcal{R}^m$, we will just concern ourselves with $\mathcal{K}$ being a 'quadrant' cone in $\mathcal{R}^m$. Define δ_i to be the discrete function $\delta_i = 1$ if $i \in I$ and $\delta_i = -1$ if $i \in J$. We can then equivalently define the cone $\mathcal{K}$ to be $\mathcal{K} = \{x \in \mathcal{R}^m \,|\, \delta_i x_i \geq 0 \text{ for } i = 1, 2, \ldots, m\}$. With this notation, the interior of $\mathcal{K}$ can be described by $\mathcal{K}^\circ = \{x \in \mathcal{R}^m \,|\, \delta_i x_i > 0, i = 1, 2, \ldots, m\}$.

We can now define the reproducing cone $\mathcal{P} \subset \mathcal{B}$ by $\mathcal{P} = \{u \in \mathcal{B} \,|\, u(t) \in \mathcal{K}, t \in [a + k, b + k]\}$ or equivalently by $\mathcal{P} = \{u \in \mathcal{B} \,|\, \delta_i u_i(t) \geq 0, i = 1, 2, \ldots, m; t \in [a + k, b + k]\}$. The interior of our cone $\mathcal{P}$ is now given in the next lemma.

LEMMA 4. *Let $\mathcal{P}$ be the cone in the Banach space $\mathcal{B}$ as defined above. The interior of $\mathcal{P}$ is given by*

$$\mathcal{P}^\circ = \{u \in \mathcal{B} \,|\, u(t) \in \mathcal{K}^\circ, t \in (a + k, b + k]\},$$

or equivalently

$$\mathcal{P}^\circ = \{u \in \mathcal{B} \,|\, \delta_i u_i(t) > 0, t \in (a + k, b + k]\}.$$

PROOF: Let $Q = \{u \in \mathcal{B} \,|\, \delta_i u_i(t) > 0, t \in (a + k, b + k]\}$. We will show that $Q = \mathcal{P}^\circ$. First let $u \in Q$. Then $\delta_i u_i(t) > 0$ for $t \in (t_i, t_n + n]$. So if we let

$\epsilon = \min_{1\leq i\leq m}\{\min_{(a+k,b+k]}|u_i(t)|\}$, then $\epsilon > 0$ since $u(t)$ is a discrete function. Now, let $y \in B(u;\epsilon)$, so $||u-y|| < \epsilon$. If $y \in \mathcal{P}$ then we are done, so assume that $y \notin \mathcal{P}$. Then, there exists a $t_\circ$ so that $y(t_\circ) \notin \mathcal{K}^\circ$ for some $t_\circ \in (a+k, b+k]$, which means that $\delta_{i_\circ} y_{i_\circ}(t_\circ) \leq 0$ for some $i_\circ \in \{1,2,\ldots,m\}$. Since $||u-y|| < \epsilon$, we have that

$$\begin{aligned}\epsilon &> |u(t_\circ) - y(t_\circ)| \\ &= \left(\sum_{i=1}^{m}(u_i(t_\circ) - y(t_\circ))^2\right)^{\frac{1}{2}} \\ &\geq |u_{i_\circ}(t_\circ) - y_{i_\circ}(t_{i_\circ})|.\end{aligned}$$

So $-\epsilon < u_{i_\circ}(t_\circ) - y_{i_\circ}(t_\circ) < \epsilon$. First, suppose $\delta_{i_\circ} = 1$. Then we have that $\delta_{i_\circ}\epsilon = \epsilon > \delta_{i_\circ} u_{i_\circ}(t_\circ) - \delta_{i_\circ} y_{i_\circ}(t_\circ) \geq u_{i_\circ}(t_\circ)$ since $\delta_{i_\circ} y_{i_\circ}(t_\circ) \leq 0$. But this contradicts the fact that $\epsilon = \min_{1\leq i\leq m}\{min_{(a+k,b+k]}|u_i(t)|\} \leq |u_{i_\circ}(t_\circ)| = u_{i_\circ}(t_\circ)$, since $\delta_{i_\circ} = 1$.

So if $\delta_{i_\circ} = 1$, we have a contradiction. Now suppose that $\delta_{i_\circ} = -1$. From above we have that $\delta_{i_\circ}(-\epsilon) > \delta_{i_\circ} u_{i_\circ}(t_\circ) - \delta_{i_\circ} y_{i_\circ}(t_\circ) > \delta_{i_\circ}\epsilon$ and from this we get that $\epsilon > \delta_{i_\circ} u_{i_\circ}(t_\circ)$ since $-\delta_{i_\circ} y_{i_\circ}(t_\circ) \geq 0$. But again this contradicts the minimality of ϵ. Hence $\delta_{i_\circ}$ is not equal to either 1 or -1 which again is a contradiction, which means that our original assumption that there exists a $t_\circ$ and a $i_\circ$ so that $\delta_{i_\circ} y_{i_\circ}(t_\circ) \leq 0$ is false. Thus $y \in \mathcal{P}$ and since y was arbitrary, we have that $B(u;\epsilon) \subset \mathcal{P}$ and so $u \in \mathcal{P}^\circ$. So we have shown that $Q \subset \mathcal{P}^\circ$.

Now let $u \in \mathcal{P}^\circ$ and we will show that $u \in Q$. Suppose $u \notin Q$, so that there exists a $t_\circ \in [a+k, b+k]$ and an $i_\circ \in \{1,2,\ldots,m\}$ so that $\delta_{i_\circ} u_{i_\circ}(t_\circ) = 0$, so $u_{i_\circ}(t_\circ) = 0$, (clearly if $\delta_{i_\circ} u_{i_\circ}(t_\circ) < 0$ then $u \notin \mathcal{P}$ which contradicts $u \in \mathcal{P}^\circ$).

Since $u \in \mathcal{P}^\circ$, there exists an ϵ so that $B(u;\epsilon) \subset \mathcal{P}$. Let $y(t)$ be such that $y(t) = u(t)$ if $t \neq t_\circ$ and when $t = t_\circ$, let $y_i(t_\circ) = u_i(t_\circ)$ for $i \neq i_\circ$ and finally, let $\delta_{i_\circ} y_{i_\circ}(t_\circ) = -\frac{\epsilon}{2}$. This gives us

$$\begin{aligned}
||u-y|| &= \max_{1\le i\le m}\{\max_{t\in(a+k,b+k]}|u_i(t)-y_i(t)|\} \\
&= \max_{1\le i\le m}\{|u_i(t_{i_\circ})-y_i(t_{i_\circ})| \\
&= |u_{i_\circ}(t_{i_\circ})-y_{i_\circ}(t_{i_\circ})| \\
&= \frac{\epsilon}{2}, \text{ since } u_{i_\circ}(t_\circ)=0
\end{aligned}$$

Hence $||u-y|| = \frac{\epsilon}{2} < \epsilon$ so we have that $y \in B(u;e) \subset \mathcal{P}$. But $\delta_{i_\circ} y_{i_\circ}(t_\circ) = -\frac{\epsilon}{2} < 0$ so $y \notin \mathcal{P}$ which is a contradiction. So we must have that $\delta_i u_i(t) > 0$ for all $1 \le i \le m$ and $t \in (a+k, b+k]$, that is $u \in Q$. Thus since we have shown that $Q \subset \mathcal{P}^\circ$ and $\mathcal{P}^\circ \subset Q$ we have that $\mathcal{P}^\circ = Q$ and our lemma is proved.

We now state our first existence result for our boundary value problem (3).

THEOREM 5. *Assume hypothesis (H) holds, $\delta_i\delta_j p_{ij}(t) \ge 0$, for $t \in [a+k, b+k]$, $1 \le i,j \le m$ and that there is a $t_\circ \in [a+k, b+k]$ and an $i_\circ$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$. Then for the eigenvalue problem (3), there exists an eigenvector $z_\circ \in \mathcal{P}$ with corresponding positive eigenvalue $\lambda_\circ$ which is a lower bound for the modulus of any other eigenvalue for this eigenvalue problem. Furthermore, $\delta_i z(t)_i \ge 0$, for all $t \in [a, b+n]$, $i = 1, 2, \ldots, m$, that is $z(t) \in \mathcal{K}$ for all $t \in [a, b+n]$.*

PROOF: We define the linear operator $M : \mathcal{B} \to \mathcal{B}$ by

$$Mu(t) = \sum_{s=a+k}^{b+k} G(t,s)P(s)u(s), \qquad \text{for } t \in [a+k, b+k],$$

where $G(t,s)$ is the Green's function for (4). We note that the eigenvalues of boundary value problem (1) are reciprocals of the operator M, and that zero is not an eigenvalue of (3) since $Ly = 0$ is right disfocal. We also note that since $G(t,s)$ is defined for all $t \in [a, b+n]$, we have that $Mu(t)$ is well defined on $[a, b+n]$.

We will now show that our compact operator M, is a positive operator, that is, $M : \mathcal{P} \to \mathcal{P}$. Let u be an arbitrary element of $\mathcal{P}$. If we can show that $\delta_i(Mu(t))_i \geq 0$ for all $t \in [t_1, t_n + n]$, $i = 1, 2, \ldots, m$, where $(Mu(t))_i$ denotes the *i-th* component of $Mu(t)$, then $Mu \in \mathcal{P}$. Consider the *i-th* component of $P(t)u(t)$, $(P(t)u(t))_i = \sum_{j=1}^m p_{ij}(t)u_j(t)$. Now $\delta_j\delta_j = 1$ and $\delta_j u_j(t) \geq 0$ so we have that for all $t \in [a+k, b+k]$, $\delta_i(P(t)u(t))_i = \sum_{j=1}^m \delta_i\delta_j p_{ij}(t)\delta_j u_j(t) \geq 0$, since $\delta_i\delta_j p_{ij}(t) \geq 0$ by hypothesis. From hypothesis (H), we have that $G(t,s) \geq 0$ on $[a, b+n] \times [a+k, b+k]$. Thus $\delta_i(Mu)_i(t) = \sum_{s=a+k}^{b+k} G(t,s) \sum_{j=1}^m \delta_i\delta_j p_{ij}(t)\delta_j u_j(t) \geq 0$, for all $t \in [a, b+n]$, $i = 1, 2, \ldots, m$. Thus $Mu \in \mathcal{P}$, and since u was an arbitrary element of $\mathcal{P}$, we have that M is a positive operator.

In order to apply Theorem 1.6, of Chapter 1, we must find a nontrivial $u_\circ \in \mathcal{P}$, and an $\varepsilon_\circ > 0$ so that $Mu_\circ \geq \varepsilon_\circ u_\circ$. Let $u_\circ(t) = \delta_{i_\circ}\epsilon_{i_\circ}$, where $\epsilon_{i_\circ}$ is the unit vector in $\mathcal{R}^m$ in the $i_\circ$ direction. This gives us that the jth component of $(u_\circ(t))_j = \delta_{i_\circ}\delta_{i_\circ j}$, where δ_{ij} is the Kronecker delta function. Thus $\delta_j(u_\circ(t))_j =$

$\{\delta_j\delta_{i_\circ}\}\delta_{i_\circ j} \geq 0$, so $u_\circ \in \mathcal{P}$. We note that $\delta_{i_\circ}(u_\circ(t))_{i_\circ} = 1 > 0$, on $[a+k, b+k]$ and that $\delta_j(u_\circ(t))_j = 0$ for all other j.

We now consider $Mu_\circ(t)$. Since $M : \mathcal{P} \to \mathcal{P}$, we know that $\delta_j(Mu_\circ)_j(t) \geq 0 = \delta_j(u_\circ(t))_j$ for $1 \leq j \leq m$, $j \neq i_\circ$. When $j = i_\circ$ we have that

$$\begin{aligned}
\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) &= \sum_{s=a+k}^{b+k} G(t,s) \sum_{j=1}^{m} \delta_{i_\circ}\delta_j p_{i_\circ j}(s)\delta_j(u_\circ(s))_j \\
&= \sum_{s=a+k}^{b+k} G(t,s)\delta_{i_\circ}\delta_{i_\circ}p_{i_\circ i_\circ}(s)\delta_{i_\circ}(u_\circ(t))_{i_\circ} \\
&= \sum_{s=a+k}^{b+k} G(t,s)p_{i_\circ i_\circ}(s) \\
&> 0, \quad \text{for } t \in [a+k, b+k],
\end{aligned}$$

since by hypothesis (H), $G(t,s) > 0$ for all $t \in (a, b+n]$, $s \in [a+k, b+k]$ and $p_{i_\circ i_\circ}(t_\circ) > 0$ for $t_\circ \in [a+k, b+k]$. So we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) > 0$ for all $t \in [a+k, b+k]$, and since $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t)$ is a discrete function, we have that $\varepsilon_\circ = \min_{[a+k,b+k]}\{\delta_\circ(Mu_\circ)_{i_\circ}(t)\} > 0$. Hence we have that $\delta_{i_\circ}(Mu_\circ)_{i_\circ}(t) \geq \varepsilon_\circ = \varepsilon_\circ(\delta_{i_\circ}(u_\circ(t))_{i_\circ})$ for $t \in [a+k, b+k]$, since $\delta_{io}(u_K(t))_{i_\circ} = 1$. This gives us that $Mu_\circ \geq \varepsilon_\circ u_\circ$ with respect to the cone $\mathcal{P}$. By applying Theorem 1.6 of Chapter 1, we have that there exists an eigenvector $z_\circ \in \mathcal{P}$ with corresponding positive eigenvalue $\lambda_\circ$ which is an upper bound for the modulus of any other eigenvalue for this eigenvalue problem. Since the eigenvalues of M are reciprocals of the eigenvalues of (3), our results follow.

We now show the final conclusions of this theorem, that is, if $(\lambda_\circ, z_\circ)$ are

the eigenpair from above, then $z_\circ(t) \in \mathcal{K}$ for all $t \in [a, b+n]$. We know that $Mz_\circ(t) = \lambda_\circ z_\circ(t)$ or $z_\circ(t) = (1/\lambda_\circ)Mz_\circ(t)$ since $\lambda_\circ > 0$. Thus,

$$\begin{aligned}
\delta_i(z_\circ(t))_i &= \delta_i(1/\lambda_\circ)(Mz_\circ(t))_i \\
&= \frac{1}{\lambda_\circ}\sum_{s=a+k}^{b+k} G(t,s)\sum_{j=1}^{m}\delta_i\delta_j p_{ij}(s)\delta_j(z_\circ(s))_j \\
&\geq 0, \qquad \text{for } t \in [a, b+n],
\end{aligned}$$

since $G(t,s) \geq 0$ for all $t \in [a, b+n]$, $s \in [a+k, b+k]$ and by hypothesis $\delta_i\delta_j p_{ij}(t) \geq 0$, for $t \in [a+k, b+k]$, $1 \leq i,j \leq m$ and $\delta_i(z_\circ(t))_i \geq 0$ since $z(t) \in \mathcal{P}$. Hence we have that $\delta_i z(t)_i \geq 0$, for all $t \in [a, b+n]$, $i = 1, 2, \ldots, m$, that is $z(t) \in \mathcal{K}$ for all $t \in [a, b+n]$.

If we have stronger conditions on $P(t)$, then we get better results.

THEOREM 6. *Assume hypothesis (H) holds, $\delta_i\delta_j p_{ij}(t) > 0$, $1 \leq i,j \leq m$, for all $t \in [a+k, b+k]$. Then for the eigenvalue (3), there exists an essentially unique eigenvector $z_\circ$ in $\mathcal{P}^\circ$, and its corresponding eigenvalue is simple, positive and smaller then the modulus of any other eigenvalue for this eigenvalue problem. Furthermore, $\delta_i z(t)_i > 0$, for all $t \in (a, b+n]$, $i = 1, 2, \ldots, m$, that is $z(t) \in \mathcal{K}^\circ$ for all $t \in (a, b+n]$.*

PROOF: As in the last proof we define the compact linear operator M by $Mu(t) = \sum_{s=a+k}^{b+k} G(t,s)P(s)u(s)$, $t_1 \leq t \leq t_n + n$. We wish to show that M is a $u_\circ$-positive operator so that we can apply Theorems 1.8 1.9 of Chapter 1. To show that M is $u_\circ$-positive, we will show that $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^\circ$, and then apply Lemma 1.5 of

Chapter 1.

Let u be an arbitrary element of $\mathcal{P}\backslash\{0\}$. Then, there exists an $i_\circ \in \{1, 2, \ldots, m\}$ and a $t_\circ \in [a+k, b+k]$ so that $\delta_{i_\circ} u_{i_\circ}(t_\circ) > 0$. By hypothesis, for each $i = 1, 2, \ldots, m$, $\delta_i \delta_{i_\circ} p_{ii_\circ}(t) > 0$ on $[a+k, b+k]$. This gives us that $\delta_i \delta_{i_\circ} p_{ii_\circ}(t) \delta_{i_\circ} u_{i_\circ}(t_\circ) > 0$ for all $t \in [a+k, b+k]$, $i = 1, 2, \ldots, m$. Then, by hypothesis (H) $G(t,s) > 0$ for all $t \in (a, b+n]$, $s \in [a+k, b+k]$, we have that for each $i = 1, 2, \ldots, m$

$$\begin{aligned}
\delta_i (Mu)_i(t) &= \sum_{s=a+k}^{b+k} G(t,s) \delta_i \sum_{j=1}^{m} p_{ij}(s) u_j(s) \\
&= \sum_{s=a+k}^{b+k} G(t,s) \sum_{j=1}^{m} \delta_i \delta_j p_{ij}(s) \delta_j u_j(s) \\
&\geq \sum_{s=a+k}^{b+k} G(t,s) \delta_i \delta_{i_\circ} p_{ii_\circ}(s) \delta_{i_\circ} u_{i_\circ}(s) \\
&> 0, \qquad \text{for } t \in (a, b+n].
\end{aligned}$$

Thus we have that $\delta_i(Mu(t))_i > 0$ for all $t \in (a, b+n]$. But this give us that $Mu(t) \in \mathcal{K}^\circ$ for all $t \in (a, b+n]$. In particular we have that $Mu(t) \in \mathcal{K}^\circ$ for all $t \in [a+k, b+k]$, and so by Theorem 4 of this chapter we have that $Mu \in \mathcal{P}^\circ$. Since u was an arbitrary, nontrivial element of $\mathcal{P}$ we have that $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^\circ$, so by Lemma 1.5 of Chapter 1 we have that M is a $u_\circ$-positive operator. Hence we now apply Theorems 1.8 and 1.9 of Chapter 1, to get that M has an essentially unique eigenvector $z_\circ$ in $\mathcal{P}^\circ$, and its corresponding eigenvalue is simple, positive and greater then the modulus of any other eigenvalue for this eigenvalue problem.

Since the eigenvalues of M are reciprocals of the eigenvalues of (3) we have our desired results.

Furthermore, from above we have that for any nontrivial $u \in \mathcal{P}$, $\delta_i(Mu(t))_i > 0$, for all $t \in (a, b+n]$, $i = 1, 2, \ldots, m$, that is $Mu(t) \in \mathcal{K}^\circ$ for all $t \in (a, b+n]$. Hence if $(\lambda_\circ, z_\circ)$ are the eigenpair for above, we have that $z_\circ(t) = (1/\lambda_\circ)Mz_\circ(t) \in \mathcal{K}^\circ$ since $\lambda_\circ > 0$ and our theorem is proven.

We also have comparison results between two focal point difference equation eigenvalue problems.

THEOREM 7. *Let hypothesis (H) hold for the eigenvalue problems (1) and (3). Also, assume that the matrix functions $P(t)$ and $Q(t)$ have the properties:*

a) There is an $i_\circ \in \{1, 2, \ldots, m\}$ and a $t_\circ \in [a+k, b+k]$ such that $p_{i_\circ i_\circ}(t_\circ) > 0$;

b) $0 \leq \delta_i\delta_j p_{ij}(t) \leq \delta_i\delta_j q_{ij}(t)$, for $t \in [a+k, b+k]$, $1 \leq i, j \leq m$;

c) $q_{ij}(t) > 0$, for $t \in [a+k, b+k]$, $1 \leq i, j \leq m$.

Then there exists smallest positive eigenvalues $\lambda_\circ$, $\Lambda_\circ$ of (1) and (3) respectively, both of which are positive, $\lambda_\circ$ a lower bound in modulus and $\Lambda_\circ$ strictly less in modulus then any other eigenvalue for their corresponding problems. If $z_\circ$ is the eigenvector corresponding to $\lambda_\circ$, then $z_\circ \in \mathcal{P}$ and in addition, $z_\circ(t) \in \mathcal{K}$ for all $t \in [a, b+n]$. Further, $\Lambda_\circ$ is a simple eigenvalue and its corresponding eigenvector, $v_\circ$ belongs to $\mathcal{P}^\circ$ and in fact, $v_\circ(t) \in \mathcal{K}$ for all $t \in (a, b+n]$.

Moreover, $\Lambda_\circ \leq \lambda_\circ$ and if $\lambda_\circ = \Lambda_\circ$, then $P(t) = Q(t)$ on $[a+k, b+k]$.

PROOF: We define the integral operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(t) = \sum_{s=a+k}^{b+k} G(t,s)P(s)u(s) \qquad \text{and} \qquad Nu(t) = \sum_{s=a+k}^{b+k} G(t,s)Q(s)u(s),$$

where $G(t,s)$ is the Green's function for (4). We know by earlier proofs that $M, N : \mathcal{P} \to \mathcal{P}$. Now, by Theorem 5, M possesses a positive eigenvalue $1/\lambda_\circ$ which is an upper bound, in modulus, for all other eigenvalues of M, and its corresponding eigenvector $z_\circ$ belongs to $\mathcal{P}$, and in addition, $z_\circ(t) \in \mathcal{K}$ for all $t \in [a, b+n]$. By Theorem 6, we have that N has a positive, simple eigenvalue $1/\Lambda_\circ$, which is strictly greater, in modulus, than all other eigenvalues of N, and its essentially unique eigenvector $v_\circ$ belongs to $\mathcal{P}^\circ$, and in fact, $v_\circ(t) \in \mathcal{K}$ for all $t \in (a, b+n]$.

We will now show that $M \leq N$ with respect to $\mathcal{P}$. Let u be an arbitrary element in $\mathcal{P}$. Then for each fixed $i \in \{1, 2, \ldots, m\}$, we have $\delta_i\delta_j(q_{ij}(t) - p_{ij}(t)) \geq 0$ for $t \in [a+k, b+k]$, $1 \leq j \leq m$. Also, since $u \in \mathcal{P}$, we know that $\delta_j(u(t))_j \geq 0$ for all $t \in [a+k, b+k]$, $1 \leq j \leq m$. These last two items and the fact that $\delta_j\delta_j = 1$ gives us that for $j = 1, 2, \ldots, m$,

$$\sum_{j=1}^{m} \delta_i(q_{ij}(t) - p_{ij}(t))(u(t))_j \geq 0$$

for $t \in [a+k, b+k]$. Now hypothesis (H) tells us that $G(t,s) \geq 0$ on $[a, b+n] \times [a+k, b+k]$, and thus

$$\delta_i \sum_{s=a+k}^{b+k} G(t,s) \sum_{j=1}^{m} (q_{ij}(t) - p_{ij}(t))(u(t))_j \geq 0.$$

Since i was arbitrary, then each component of $\sum_{s=a+k}^{b+k} G(t,s)(Q_{ij}(t) - P_{ij}(t))u(t)$ times δ_i is nonnegative for all $t \in [a+k, b+k]$. Thus $\delta_i \sum_{s=a+k}^{b+k} G(t,s)(Q_{ij}(t) - P_{ij}(t))u(t) = (N-M)u(t) \in \mathcal{K}$ for all $t \in []$. Thus $Nu \geq Mu$ with respect to the cone $\mathcal{P}$. Since u was an arbitrary element of $\mathcal{P}$, we have that $M \leq N$.

Now $(\frac{1}{\lambda_\circ}, z_\circ)$ and $(\frac{1}{\Lambda_\circ}, v_\circ)$ are eigenpairs of M and N respectively, so we have that the inequalities of Theorem 1.11, Chapter 1, hold. Also, similiar to the proof in Theorem 6, we have that N is $u_\circ$-positive. From above we see that $M \leq N$, and so we can apply Theorem 1.11, Chapter 1 to give us that $\frac{1}{\lambda_\circ} \leq \frac{1}{\Lambda_\circ}$ or $\Lambda_\circ \leq \lambda_\circ$.

Finally, suppose that $\lambda_\circ = \Lambda_\circ \doteq \lambda$, then Theorem 1.11, Chapter 1 tells us that $z_\circ = kv_\circ$ for some nonzero scalar k. Then $\lambda P(t)z_\circ = Lz_\circ = kLv_\circ = k\lambda Q(t)v_\circ = \lambda Q(t)z_\circ$. Thus $\lambda P(t)z_\circ = \lambda Q(t)z_\circ$ so $(Q(t) - P(t))z_\circ = 0$ since $\lambda \neq 0$. So, for each i-th component of $(Q(t) - P(t))z_\circ = 0$,

$$\sum_{j=1}^{m}(q_{ij}(t) - p_{ij}(t)(z_\circ(t))_j = 0, \quad \text{for } t \in [a+k, b+k].$$

Hence

$$\sum_{j=1}^{m}[\delta_i\delta_j(q_{ij}(t) - p_{ij}(t)]\delta_j(z_\circ(t))_j = 0, \quad \text{for } t \in [a+k, b+k],$$

and since $z_\circ \in \mathcal{P}^\circ$ we have that $\delta_j z_\circ(t) > 0$ for all $t \in [a+k, b+k]$. This, plus the fact that $\delta_i\delta_j q_{ij}(t) \geq \delta_i\delta_j p_{ij}(t)$ for $t \in [a+k, b+k]$, $1 \leq i,j \leq m$, gives us that

$$p_{ij}(t) = q_{ij}(t), \quad \text{for all } t \in [a+k, b+k], \quad 1 \leq i,j \leq m.$$

Thus we have that $P(t) = Q(t)$ on the interval $[a+k, b+k]$.

V) EXAMPLES

In our final section, we will give examples for which hypothesis (H) holds.

Example n=2:

In this example we have $k = 1$, and $Lu(t) = u(t+2)+p_1(t)u(t+1)+p_2(t)u(t)$. Let $t_1 = a$ and $t_2 = b + 1$ be elements of any interval $[a, b]$ over which L is right disfocal. Then, from Theorem 3, our Green's function for (4) is, for $t \in [t_1, t_2 + 1]$, $s \in [t_1 + 1, t_2]$

$$G(t,s) = \begin{cases} \frac{-1}{\Delta y_1(t_2,t_1)} \begin{vmatrix} 0 & y_1(t,t_1) \\ \Delta y_1(t_2,s) & \Delta y_1(t_2,t_1) \end{vmatrix} & t < s+1, \\ \frac{-1}{\Delta y_1(t_2,t_1)} \begin{vmatrix} y_1(t,s) & y_1(t,t_1) \\ \Delta y_1(t_2,s) & \Delta y_1(t_2,t_1) \end{vmatrix} & s \leq t. \end{cases}$$

To show that hypothesis (H) holds for this example, we will need a difference equation analog of Rolle's Theorem, which is provided by Hartman [9].

PROPOSITION. *Suppose that $y(t)$ has N generalized zeros on $[a, b]$ and that $\Delta y(t)$ has M generalized zeros on $[a, b-1]$. Then $M \geq N - 1$.*

Now consider $y_1(t, s)$ for any $t \in [t_1, t_2 + 1]$, $s \in [t_1 + 1, t_2]$. We know that $y_1(s, s) = 0$ and $y_1(t, s) \neq 0$ for all $t \neq s$ or else by the preceding proposition we contradict $Ly = 0$ is right disfocal. Thus since $\Delta y_1(s, s) = 1$, we know $y_1(t, s) < 0$ for all $t < s$ and $y_1(t, s) > 0$ for all $t > s$. We also know that $\Delta y_1(t, s) \neq 0$ for all $t > s$ or else we again have a contradiction. Thus we have that $\Delta y_1(t, s) > 0$ for all $t > s$.

Then, when $t < s+1$ we have that

$$\begin{aligned} G(t,s) &= \frac{-1}{\Delta y_1(t_2,t_1)}\{-\Delta y_1(t_2,s)y_1(t,t_1)\} \\ &= \frac{\Delta y_1(t_2,s)y_1(t,t_1)}{\Delta y_1(t_2,t_1)}. \end{aligned}$$

So $G(t,s) \geq 0$ on $[t_1, s+1)$ and positive when $t_1 < t < s+1$.

Now suppose that $t \geq s$. Then

$$\begin{aligned} G(t,s) &= \frac{-1}{\Delta y_1(t_2,t_1)}\{y_1(t,s)\Delta y_1(t_2,t_1) - \Delta y_1(t_2,s)y_1(t,t_1)\} \\ &= \frac{\Delta y_1(t_2,s)y_1(t,t_1) - y_1(t,s)\Delta y_1(t_2,t_1)}{\Delta y_1(t_2,t_1)}. \end{aligned}$$

From the previous case we know that $G(s,s) > 0$. Suppose we define $z(t)$ to be $z(t) = \Delta y_1(t_2,s)y_1(t,t_1) - y_1(t,s)\Delta y_1(t_2,t_1)$. Then $z(s) > 0$ and $\Delta z(t_2) = 0$. Then since $z(t)$ is a solution of $Ly = 0$, we must have that $z(t) > 0$ for $t \in [s, t_2]$. Further, $\Delta z(t_2) = z(t_2+1) - z(t_2) = 0$ and so $z(t_2+1) > 0$.

Thus $G(t,s) > 0$ for $t \in (t_1, t_2+1]$, $s \in [t_1+1, t_2]$ and hence we have that when $n = 2$, hypothesis (H) holds over any interval on which $Ly = 0$ is right disfocal.

In our next two examples we will take L to be $Ly = \Delta^n y$. We note here that when $Ly = \Delta^n y$, then $Ly = 0$ is right disfocal over any interval I.

We will now need what is known as the *factoral function*. This function, $t^{(k)}$, is defined as follows:

a) if $k = 1, 2, 3, \ldots$, then $t^{(k)} = t(t-1)(t-2)\ldots(t-k+1)$;

b) if $k = 0$, then $t^{(0)} = 1$;

c) if $k = -1, -2, -3, \ldots$, then $t^{(k)} = \frac{1}{(t+1)(t+2)\ldots(t-k)}$;

d) if k is not an integer, then $t^{(k)} = \frac{\Gamma(t+1)}{\Gamma(t-k+1)}$, where $\Gamma(t)$ is the gamma function.

It is understood that the definition of $t^{(k)}$ is given only for those values of t and k which make the formula meaningful. We note that for k a positive integer, we have that

$$\begin{aligned}
\Delta t^{(k)} &= (t+1)^{(k)} - t^{(k)} \\
&= (t+1)t^{(k-1)} - t(t-1)\ldots(t-k+1) \\
&= (t+1)t^{(k-1)} - t^{(k-1)}(t-k+1) \\
&= k\,t^{(k-1)}.
\end{aligned}$$

So $\Delta t^{(k)} = kt^{(k-1)}$. This property and induction gives us, for j an integer, if $j < k$, then $\Delta^j t^{(k)} = k(k-1)\ldots(k-j+1)t^{(k-j)}$; if $j = k$ then $\Delta^k t^{(k)} = k!$; and if $j > k$ then $\Delta^j t^{(k)} = 0$. Let s be a fixed element of $[t_1 + k, t_n + k]$ and define $y_k(t,s) \doteq (t-s)^{(k)}/k!$, for each $k = 1, 2, \ldots, n-1$. Then y_k is a solution to the intitial value problem $Ly = 0$, $\Delta^j y_k = \delta_{jk}$, $0 \leq j \leq n-1$. With this in mind, we will take our set of n linearly independent solutions to $Ly = 0$ to be $\{1, y_1(t,s), \ldots, y_{n-1}(t,s)\}$.

This will simplify our Green's function considerably, since by the properties of the factoral function we have that $\Delta^j y_k = y_{k-j}$, for $j \leq k$ and $\Delta^j y_k = 0$ for $j > k$. Further, this gives us, like in Chapter 3, that $D = 1$.

Example $n = 3$:

When $n = 3$ our difference equation is $(-1)^{n-1}Ly = 0$ $Ly = \Delta^3 y$ with boundary conditions $y(t_1) = \Delta y(t_2) = \Delta^2 y(t_3) = 0$, where $a = t_1 < t_2 < t_3 = b+1$. We will show, that Hypothesis (H) holds under the condition that $(t_2 - t_1) > (t_3 - t_2)$. It is not too difficult to show that if $(t_2 - t_1) \leq (t_3 - t_2)$, then hypothesis (H) does not hold.

From Lemma 2 we have that for this equation, our Green's function for $t \in [t_1, t_3 + 2]$, is

$$G(t,s) = \begin{cases} \text{for } s \in [t_1 + k, t_2 + k - 1] & \\ \begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) \\ y_1(t_2, s-k+1) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t < s - k + 3, \\ \begin{vmatrix} y_2(t, s-k+1) & y_1(t,t_1) & y_2(t,t_1) \\ y_1(t_2, s-k+1) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & s - k + 1 \leq t, \\ \text{for } s \in [t_2 + k - 1, t_3 + k - 1] & \\ \begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & t < s - k + 3, \\ \begin{vmatrix} y_2(t, s-k+1) & y_1(t,t_1) & y_2(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} & s - k + 1 \leq t. \end{cases}$$

For Hypothesis (H) we need to show that $G(t,s) > 0$ for $t \in (t_1, t_3 + 2]$, $s \in [t_1 + k, t_3 + k - 1]$. We will first show that $\Delta G(t_1, s) > 0$ for $s \in [t_1 + k, t_3 + k - 1]$.

First, let $s \in [t_1 + k, t_2 + k - 1]$. Then we have

$$\Delta G(t_1, s) = \begin{vmatrix} 0 & \Delta y_1(t_1, t_1) & \Delta y_2(t_1, t_1) \\ y_1(t_2, s-k+1) & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 \\ y_1(t_2, s-k+1) & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} y_1(t_2, s-k+1) & y_1(t_2, t_1) \\ 1 & 1 \end{vmatrix} = y_1(t_2, t_1) - y_1(t_2, s-k+1)$$

$$= (t_2 - t_1) - (t_2 - (s-k+1)) = s - k + 1 - t_1$$

$$\geq (t_1 + k) - k + 1 - t_1 = 1 > 0.$$

If we have that $s \in [t_2 + k - 1, t_3 + k - 1]$, then

$$\Delta G(t_1, s) = \begin{vmatrix} 0 & \Delta y_1(t_1, t_1) & \Delta y_2(t_1, t_1) \\ 0 & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & y_1(t_2, t_1) \end{vmatrix} = y_1(t_2, t_1) = (t_2 - t_1) > 0.$$

Thus for $s \in [t_1 + k, t_3 + k - 1]$ we have that $\Delta G(t_1, s) > 0$.

We will now show why the condition $(t_2 - t_1) > (t_3 - t_2)$ will insure us that $G(t, s) > 0$ for $t \in (t_1, t_3 + 2]$, $s \in [t_1 + k, t_3 + k - 1]$. We have two cases to consider, when $s \in [t_1 + k, t_2 + k - 1]$ and $s \in [t_2 + k - 1, t_3 + k - 1]$.

Case 1) Fix $s \in [t_1 + k, t_2 + k - 1]$.

If $t \in (t_1, s - k + 3)$ then we have $G(t_1, s) = 0$, $\Delta G(t_1, s) > 0$ and

$$\Delta^2 G(t, s) = \begin{vmatrix} 0 & 0 & 1 \\ y_1(t_2, s-k+1) & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix} = -1.$$

So $\Delta^2 G(t,s) < 0$ and hence $\Delta G(t,s)$ is a decreasing function on $(t_1, s-k+3)$. But $G(t_1,s) = 0$ and $\Delta G(t_1,s) > 0$. Thus, if $G(s,s) > 0$, then $G(t,s) > 0$ for all $t \in (t_1, s-k+3)$. Now, if we can show that $G(t,s) > 0$ for all $t \in [s-k+1, t_3+2]$ then we will have that $G(t,s) > 0$ for all $t \in (t_1, t_3+3]$, s fixed in $[t_1+k, t_2+k-1]$.

Let $t \in [s-k+1, t_3+2]$ and define $f(t)$ on $[t_1, t_3+2]$ by

$$f(t) = \begin{vmatrix} y_2(t, s-k+1) & y_1(t,t_1) & y_2(t,t_1) \\ y_1(t_2, s-k+1) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix}.$$

Now $f(t) \equiv G(t,s)$ for $t \in [s-k+1, t_3+2]$. Thus, $f(t)$ is a solution to our differential equation $Ly = 0$ and satisfies the boundary conditions $\Delta f(t_2) = 0$ and $\Delta^2 f(t_3) = 0$. Since $\Delta^3 f(t) = 0$, $\Delta^2 f(t)$ is equal to a constant. But $\Delta^2 f(t_3) = 0$ so $\Delta^2 f(t) \equiv 0$ and so $\Delta f(t)$ is equal to a constant. But $\Delta f(t_2) = 0$ so $\Delta f(t) \equiv 0$. Thus $f(t)$ is equal to a constant on $[t_1, t_3+2]$. Evaluating $f(t)$ at t_1 gives us

$$f(t_1) = \begin{vmatrix} y_2(t_1, s-k+1) & y_1(t_1,t_1) & y_2(t_1,t_1) \\ y_1(t_2, s-k+1) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} y_2(t_1, s-k+1) & 0 & 0 \\ y_1(t_2, s-k+1) & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix} = y_2(t_1, s-k+1).$$

So $f(t) = y_2(t_1, s-k+1) = (t_1 - (s-k+1))^{(2)}/2! = \frac{1}{2}(t_1 - s + k - 1)(t_1 - s + k - 2)$. Now $(t_1 - s + k - 2) < (t_1 - s + k - 1) < (t_1 - (t_1 + k - 1) + k - 1) = 0$, since $s > t_1 + k - 1$. Thus $(t_1 - s + k - 2) < (t_1 - s + k - 1) < 0$ so $f(t_1) = y_2(t_1, s-k+1) > 0$. Thus $G(t,s) > 0$ for $t \in [s-k+1, t_3+2]$ when $s \in [t_1+k, t_2+k-1]$. So we have that when $s \in [t_1+k, t_2+k-1]$, $G(t,s) > 0$ for all $t \in (t_1, t_3+2]$.

Case 2) Fix $s \in [t_2 + k - 1, t_3 + k - 1]$.

When $t < s - k + 3$, we have that $G(t_1, s) = 0$, $\Delta G(t_1, s) > 0$, and, like in Case 1, $\Delta^2 G(t, s) = -1$. So, like before, we only need to consider $G(t, s)$ when $t \in [s - k + 1, t_3 + 2]$.

Let $t \in [s - k + 1, t_3 + 2]$ and define

$$f(t) = \begin{vmatrix} y_2(t, s-k+1) & y_1(t, t_1) & y_2(t, t_1) \\ 0 & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

for $t \in [t_1, t_3 + 3]$. So $f(t) \equiv G(t, s)$ when $t \in [s - k + 1, t_3 + 3]$. Again we know that $\Delta^3 f(t) = 0$ and that $\Delta^2 f(t_3) = 0$. Thus $\Delta f(t)$ is a constant. Evaluating $\Delta f(t)$ at $s - k + 1$ we have

$$\begin{aligned} \Delta f(s-k+1) &= \begin{vmatrix} y_1(s-k+1, s-k+1) & 1 & y_1(s-k+1, t_1) \\ 0 & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 & y_1(s-k+1, t_1) \\ 0 & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix} \\ &= y_1(t_2, t_1) - y_1(s-k+1, t_1) \\ &= (t_2 - t_1) - ((s-k+1) - t_1) \\ &= t_2 - s + k - 1 \leq t_2 - (t_2 + k - 1) + k - 1 = 0. \end{aligned}$$

Thus $\Delta f(t) \leq 0$ so $f(t)$ is non-increasing on $[t_1, t_3 + 2]$. So if $f(t_3 + 2) > 0$ then we would have what we want, $0 < f(t) = G(t, s)$ for $t \in [s - k + 1, t_3 + 2]$. If we

expand $f(t_3+2)$ along the first column, we get

$$f(t_3+2) = \begin{vmatrix} y_2(t_3+2, s-k+1) & y_1(t_3+2, t_1) & y_2(t_3+2, t_1) \\ 0 & 1 & y_1(t_2, t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= y_2(t_3+2, s-k+1) + \begin{vmatrix} y_1(t_3+2, t_1) & y_2(t_3+2, t_1) \\ 1 & y_1(t_2, t_1) \end{vmatrix}$$

$$= y_2(t_3+2, s-k+1) + \{y_1(t_3+2, t_1)y_1(t_2, t_1) - y_2(t_3+2, t_1)\}$$

$$= \frac{(t_3+2-(s-k+1))^{(2)}}{2!} + \left\{(t_3+2-t_1)(t_2-t_1) - \frac{(t_3+2-t_1)^{(2)}}{2!}\right\}$$

$$= \frac{(t_3+1-s+k)^{(2)}}{2!} + \left\{(t_3+2-t_1)(t_2-t_1) - \frac{(t_3+2-t_1)(t_3+1-t_1)}{2!}\right\}$$

$$= \frac{(t_3+1-s+k)^{(2)}}{2!} + \frac{(t_3+2-t_1)}{2!}\{(t_2-t_1) - (t_3-t_2) - 1\}.$$

Consider the first quantity, $\frac{1}{2}(t_3+1-s+k)^{(2)} = \frac{1}{2}(t_3+1-s+k)(t_3-s+k)$. Now $s \le t_3+k-1$ and so $(t_3+1-s+k) > (t_3-s+k) \ge (t_3-(t_3+k-1)+k) = 1$. This gives us that $\frac{1}{2}(t_3+2-(s-k+1))^{(2)} > 0$.

Since the first quantity is greater than zero, we only need to have the second quantity greater than or equal to zero. That is, we need $\frac{1}{2}(t_3+2-t_1)\{(t_2-t_1) - (t_3-t_2) - 1\} \ge 0$. This will occur if $\{(t_2-t_1)-(t_3-t_2)-1\} > 0$, that is, if $(t_2-t_1) \ge (t_3-t_2)+1$. Thus, since $(t_2-t_1) > (t_3-t_2)$, we have that $f(t) > 0$ and so $G(t,s) > 0$ for all $t \in (t_1, t_3+2]$ and $s \in [t_2+k-1, t_3+k-1]$.

Hence, combining the two cases, we have that for the boundary value problem, $\Delta^3 y(t-k) = 0$ and $Ty = 0$, hypothesis (H) holds provided that $(t_2-t_1) > (t_3-t_2)$.

Example $n = 4$:

In our final example of this chapter, we will take our difference equation to be $(-1)^{n-1}Ly(t) = -\Delta^4 y(t-k) = 0$, with boundary conditions $\Delta^{i-1}y(t_i) = 0$, for $i = 1, 2, 3$ and 4. We will show that Hypothesis (H) holds, under the conditions $(t_2 - t_1) > (t_4 - t_2) + 1$ and $(t_3 - t_2) > (t_4 - t_3) + 1$. For our difference equation when $n = 4$ we have from Lemma 2 that the Green's function for $t \in [t_1, t_4 + 3]$, is

$$G(t,s)=\begin{cases} \text{for } s\in[t_1+k,t_2+k-1] & \\ -\begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ y_2(t_2,s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t<s-k+4 \\ -\begin{vmatrix} y_3(t,s-k+1) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ y_2(t_2,s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & s-k+1\le t \\ \text{for } s\in[t_2+k-1,t_3+k-1] & \\ -\begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t<s-k+4 \\ -\begin{vmatrix} y_3(t,s-k+1) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & s-k+1\le t \\ \text{for } s\in[t_3+k-1,t_4+k-1] & \\ -\begin{vmatrix} 0 & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & t<s-k+4 \\ -\begin{vmatrix} y_3(t,s-k+1) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} & s-k+1\le t. \end{cases}$$

We will first show that $\Delta G(t_1,s)>0$ for all $s\in(t_1+k,t_4+k-1)$. In all cases, consider the first row of $\Delta G(t_1,s)$,

$\Delta\vec{\mathbf{R}}_1(t_1) = (0,\Delta y_1(t_1,t_1),\Delta y_2(t_1,t_1),\Delta y_3(t_1,t_1)) = (0,1,0,0)$. If we expand

$\Delta G(t_1, s)$ along the first row, we have

$$\Delta G(t_1,s) = \begin{cases} \text{for } s \in [t_1+k, t_2+k-1] \\ \begin{vmatrix} y_2(t_2, s-k+1) & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3, s-k+1) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix}, \\ \text{for } s \in [t_2+k-1, t_3+k-1] \\ \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3, s-k+1) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix}, \\ \text{for } s \in [t_3+k-1, t_4+k-1] \\ \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix}. \end{cases}$$

If we consider $\Delta G(t_1, s)$ as a function of s, we can define functions $h_i(s)$ on $[t_1+k-1, t_4+k-1]$ for $i = 1,2,3$ to be

$$h_1(s) = \begin{vmatrix} y_2(t_2, s-k+1) & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3, s-k+1) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} \text{ for } t_1+k-1 \le s \le t_4+k-1$$

$$h_2(s) = \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3, s-k+1) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} \text{ for } t_1+k-1 \le s \le t_4+k-1$$

$$h_3(s) = \begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix} \text{ for } \quad t_1+k-1 \le s \le t_4+k-1.$$

Then $h_i(s) = \Delta G(t_1, s)$ when $s \in I_i$, for $i = 1,2,3$. By defining the h_i's in this manner we have that $h_1(t_2+k-1) = h_2(t_2+k-1)$ and $h_2(t_3+k-1) = h_3(s)$, since h_3 is a constant function. We will need to take the difference, with respect to s, of these functions and will denote this operator by Δ_s. We note without proof that $\Delta_s^j y_k(t,s) = (-1)^j y_{k-j}(t,s)$ if $k \ge j$ and zero otherwise.

Now $h_1(t_1+k-1)=0$ since, in this determinant, the first and last columns are equal. Also, we have that $\Delta_s h_1(s) = -\begin{vmatrix} y_1(t_2, s-k+1) & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 1 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 1 \end{vmatrix}$, so that $\Delta_s h_1(t_1+k-1)=0$ since, in this case, the first and second columns are equal. Finally, $\Delta_s^2 h_1(s) = \begin{vmatrix} 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 1 \end{vmatrix} = 1$. The last equation gives us that $\Delta_s h_1$ is increasing on $[t_1+k-1, t_4+k-2]$. Now $\Delta_s h_1(t_1+k-1)=0$ so $\Delta_s h_1 > 0$ on $[t_1+k, t_4+k-2]$. So h_1 in increasing on this interval and $h_1(t_1+k-1)=0$. Thus we have shown, in particular, that $h_1(s)>0$ for all s in $[t_1+k, t_2+k-1]$.

Now $\Delta_s h_2(s) = -\begin{vmatrix} 0 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 1 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 1 \end{vmatrix} = y_1(t_2,t_1) = (t_2-t_1) > 0$. So h_2 is an increasing function with $h_2(t_2+k-1) = h_1(t_2+k-1) > 0$. Thus h_2 is positive on $[t_2+k-1, t_3+k-1]$.

Finally, h_3 is constant and $h_3(s) = h_2(t_3+k-1) > 0$. So h_3 is positive on $[t_3+k-1, t_4+k-1]$. Putting this all together we have that $\Delta G(t_1,s) > 0$ for all s in $[t_1+k, t_4+k-1]$.

We will now show why the conditions $(t_2-t_1) > (t_4-t_2)+1$ and $(t_3-t_2) > (t_4-t_3)+1$ insure us that $G(t,s)>0$ for $t \in (t_1, t_4+3]$, $s \in (t_1+k, t_4+k-1)$. We have three cases to consider.

<u>Case 1</u>: Fix $s \in [t_1+k, t_2+k-1]$

For $t_1 < t \le s-k+1 \le s-k+4$, we have that

$$\Delta^2 G(t,s) = -\begin{vmatrix} 0 & 0 & 1 & y_1(t,t_1) \\ y_2(t_2,s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 1 & y_1(t,t_1) \\ y_1(t_3,s-k+1) & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= y_1(t_3,s-k+1) - \{y_1(t_3,t_1) - y_1(t,t_1)\}$$

$$= (t_3 - (s-k+1)) - (t_3 - t_1) + (t - t_1)$$

$$= t - (s-k+1) \le 0 \quad \text{since } t \le s-k+1.$$

So $\Delta^2 G(t,s) \le 0$ on $(t_1, s-k+1]$, and hence $\Delta G(t,s)$ is a decreasing function on $(t_1, s-k+1]$. Now $G(t_1,s) = 0$ and we have previously shown that $\Delta G(t_1,s) > 0$. Thus if $G(s-k+1,s) > 0$, then $G(t,s) > 0$ for all t in $(t_1, s-k+1]$. So we now consider $G(t,s)$ for $t \in [s-k+1, t_4+3]$ and will show that it is positive.

Now, for fixed s, $G(t,s)$ is a solution of $Ly = 0$ so $\Delta^4 G(t,s) = 0$. Thus $\Delta^3 G(t,s)$ is a constant. But $\Delta^3 G(t_4,s) = 0$ since $G(t,s)$ satisfies the boundary conditions. So $\Delta^3 G(t,s) \equiv 0$ and so $\Delta^2 G(t,s)$ is a constant. Now $\Delta^2 G(t_3,s) = 0$ and so $\Delta^2 G(t,s) \equiv 0$. This gives us that $\Delta G(t,s)$ is a constant. But $\Delta G(t_2,s) = 0$ so $\Delta G(t,s) \equiv 0$. Thus $G(t,s)$ is a constant on $[s-k+1, t_4+3]$.

Now, define $f(t)$ on $[t_1, t_4+3]$ by

$$f(t) = -\begin{vmatrix} y_3(t,s-k+1) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ y_2(t_2,s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}.$$

So we have that $f(t) = G(t,s)$ when $t \in [s-k+1, t_4+4]$. Evaluating f at t_1 gives us

$$f(t_1) = -\begin{vmatrix} y_3(t_1, s-k+1) & y_1(t_1,t_1) & y_2(t_1,t_1) & y_3(t_1,t_1) \\ y_2(t_2, s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3, s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} y_3(t_1, s-k+1) & 0 & 0 & 0 \\ y_2(t_2, s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3, s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -y_3(t_1, s-k+1) = -\frac{(t_1-(s-k+1))^{(3)}}{3!}$$

$$= -\frac{1}{3!}(t_1-s+k-1)(t_1-s+k-2)(t_1-s+k-3).$$

Now, $(t_1-s+k-3) < (t_1-s+k-2) < (t_1-s+k-1) \leq (t_1-(t_1+k)+k-1) = -1 < 0$, since $s \in [t_1+k, t_2+k-1]$. Thus $f(t_1) = -\frac{1}{3!}(t_1-(s-k+1))^{(3)} > 0$. Since $f(t) = G(t,s)$ on $[s-k+1, t_4+3]$ and $G(t,s)$ is constant, we have that $G(t,s) > 0$ on $[s-k+1, t_4+3]$. But, as noted earlier, this implies that $G(t,s) > 0$ for all $t \in (t_1, t_4+3]$ when $s \in [t_1+k, t_2+k-1]$.

<u>Case 2</u>: Fix $s \in [t_2+k-1, t_3+k-1]$. Let $t \in (t_1, s-k+1]$ and consider $\Delta^2 G(t,s)$. As in Case 1, we have that $\Delta^2 G(t,s) = t-(s-k+1) \leq 0$. This can be easily seen since the only difference between this expression and the one in Case 1, is the element $y_2(t_2, s-k+1)$, which lies in the second row, first column slot. After taking two differences of $G(t,s)$, we will expand along the second column, which has only one nonzero element, in the second slot. This will eliminate the $y_2(t_2, s-k+1)$ term, and $\Delta^2 G(t,s)$ will be the same as in Case 1.

Thus $\Delta^2 G(t,s) \leq 0$, since $t \leq s-k+1$. Since $G(t_1,s)=0$ and $\Delta G(t_1,s) > 0$, we only have to evaluate $G(t,s)$ at $t = s-k+1$. Similar to the last case, we will now show that $G(t,s) > 0$ for $t \in [s-k+1, t_4+3]$.

We now let $t \in [s-k+1, t_4+3]$. We know that $G(t,s)$ is a solution to $\Delta^4 y = 0$ on $(s-k+4, t_4+3]$ and satisfies the appropriate boundary conditions. So we have that $\Delta^4 G(t,s) \equiv 0$ and $\Delta^3 G(t_4,s) = \Delta^2 G(t_3,s) = 0$. This gives us that $\Delta G(t,s)$ is a constant function. Define the function $f(t)$ on $[t_1, t_4+3]$, to be

$$f(t) = -\begin{vmatrix} y_3(t,s-k+1) & y_1(t,t_1) & y_2(t,t_1) & y_3(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}.$$

So $f(t) \equiv G(t,s)$ when $t \geq s-k+1$. This then gives us that $\Delta f(t)$ is a constant function. Evaluating Δf at t_2 and using properties of determinants we have

$$\begin{aligned}\Delta f(t_2) &= -\begin{vmatrix} y_2(t_2,s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} \\ &= -\begin{vmatrix} y_2(t_2,s-k+1) & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &\qquad - \begin{vmatrix} 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ y_1(t_3,s-k+1) & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix} \\ &= -y_2(t_2,s-k+1) = -\frac{(t_2-(s-k+1))^{(2)}}{2!}.\end{aligned}$$

Now $(t_2-(s-k+1))^{(2)} = (t_2-s+k-1)(t_2-s+k-2)$ and since $s \geq t_2+k-1$ we

have that $(t_2 - s - k - 2) < (t_2 - s + k - 1) \leq (t_2 - (t_2 + k - 1) + k - 1) = 0$. Thus $\Delta f(t_2) \leq 0$ and since $\Delta f(t)$ is constant, we have that $\Delta f(t) = \Delta G(t,s) \leq 0$. Thus $G(t,s)$ is a nonincreasing function for $t \in [s-k+1, t_4+3]$. So if $G(t_4+4, s) > 0$, then we would have that $G(t,s) > 0$ for all t in $(t_1, t_4+3]$.

Consider $\Delta_s G(t_4+3, s)$ as a function in s for $s \in [t_2+k-1, t_3+k-2]$. We have

$$\Delta_s G(t_4+3,s) = -\begin{vmatrix} -y_2(t_4+3,\hat{s}) & y_1(t_4+3,t_1) & y_2(t_4+3,t_1) & y_3(t_4+3,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ -1 & 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} y_2(t_4+3, s-k+1) & y_1(t_4+3,t_1) & y_2(t_4+3,t_1) \\ 0 & 1 & y_1(t_2,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= y_2(t_4+3, s-k+1) + \{y_1(t_4+3,t_1)y_1(t_2,t_1) - y_2(t_4+3,t_1)\}$$

$$= \frac{(t_4+3-(s-k+1))^{(2)}}{2!} + \left\{(t_4+3-t_1)(t_2-t_1) - \frac{(t_4+3-t_1)^{(2)}}{2!}\right\},$$

where $\hat{s} = s-k+1$ in the first determinant. Now $s \in [t_2+k-1, t_3+k-2]$ so $(t_4+3-(s-k+1))^{(2)} = (t_4+2-s+k)(t_4+1-s+k) > 0$, since $(t_4+1-s+k) \geq (t_4+1-(t_3+k-2)+k) = (t_4-t_3+3) > 0$. So our first term is positive. The second term is nonnegative since

$$\{(t_4+3-t_1)(t_2-t_1)-\frac{1}{2!}(t_4+3-t_1)^{(2)}\}$$
$$=\{(t_4+3-t_1)(t_2-t_1)-\frac{1}{2!}(t_4+3-t_1)(t_4+2-t_1)\}$$
$$=\frac{1}{2!}(t_4+3-t_1)\{2(t_2-t_1)-(t_4+2-t_1)\}$$
$$=\frac{1}{2!}(t_4+3-t_1)\{(t_2-t_1)-(t_4-t_2)-2\}$$
$$\geq 0,$$

since we required that $(t_2-t_1)>(t_4-t_2)+1$. Thus $\Delta_s G(t,s)>0$ for all elements $s\in[t_2+k-1,t_3+k-2]$. This tells us that $G(t,s)$ is an increasing function in s, and so $G(t_4+3,t_2+k-1)\leq G(t_4+3,s)$ for all $s\in[t_2+k-1,t_3+k-1]$. But in our previous case we proved that $G(t,s)>0$ for all $t\in(t_1,t_4+3]$, $s\in(t_1+k,t_2+k-1]$. Thus, $0<G(t_4+3,t_2+k-1)\leq G(t_4+3,s)$ for all $s\in[t_2+k-1,t_3+k-1]$ provided that $(t_2-t_1)>(t_4-t_2)+1$.

Summing up, we have shown that if $(t_2-t_1)>(t_4-t_2)+1$, then $G(t,s)>0$ for all t in $(t_1,t_4+3]$, s fixed in $[t_2+k-1,t_3+k-1]$.

Our final case is when s is an element of $[t_3+k-1,t_4+k-1)$.

Case 3: Fix $s\in[t_3+k-1,t_4+k-1]$.

Let $t \leq s - k + 1$ and consider $\Delta^2 G(t,s)$ which is

$$\Delta^2 G(t,s) = -\begin{vmatrix} 0 & 0 & 1 & y_1(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & y_1(t,t_1) \\ 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \end{vmatrix} = -\begin{vmatrix} 1 & y_1(t,t_1) \\ 1 & y_1(t_3,t_1) \end{vmatrix}$$

$$= y_1(t,t_1) - y_1(t_3,t_1) = (t - t_1) - (t_3 - t_1)$$

$$= (t - t_3).$$

This gives that $\Delta^2 G(t,s) \leq 0$ on $[t_1, t_3]$ and $\Delta^2 G(t,s) \geq 0$ on $[t_3, s-k+1]$. Since $G(t_1, s) = 0$ and $\Delta G(t_1, s) > 0$, all we have to worry about is the sign of $G(t,s)$ for $t \in [t_3, s+k-1]$. We know $\Delta G(t_2, s) = 0$ and $\Delta^2 G(t,s) = t - t_3$, so $\Delta G(t,s) < 0$ on $(t_2, t_3]$ and then begins to increase. Now, as it turns out, $\Delta G(t_\circ, s) = 0$ where $t_\circ = t_3 + (t_3 - t_2) + 1$. This can be verified by direct substitution, but the algebra is exhaustive. (For a motivation of why this $t_\circ$ works, see example 4 of Chapter 3.) Hence, for $t \in (t_2, t_\circ)$ we have that $\Delta G(t,s) < 0$.

Now $t \leq s - k + 1 \leq (t_4 + k - 1) - k + 1 = t_4$. Thus $t \leq t_4 < t_4 + 2 < t_3 + (t_3 - t_2) + 1 = t_\circ$, since by hypothesis we have $(t_4 - t_3) + 1 < (t_3 - t_2)$. Hence $\Delta G(t,s) < 0$ on $(t_2, s-k+1]$ and so we have that $G(t,s)$ is a decreasing function on $(t_2, s+k-1]$. This gives us that if $G(s+k-1, s) > 0$, then $G(t,s) > 0$ for all $t \in (t_1, s+k-1]$. Like before, we will now show that $G(t,s) > 0$ for all $t \in [s+k-1, t_4+3]$.

Let t be in the interval $[s+k-1, t_4+3]$ and consider $\Delta^2 G(t,s)$ on $[s+k-1, t_4+1]$,

$$\Delta^2 G(t,s) = -\begin{vmatrix} y_1(t,s-k+1) & 0 & 1 & y_1(t,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} y_1(t,s-k+1) & 1 & y_1(t,t_1) \\ 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 1 \end{vmatrix}$$

$$= -y_1(t,s-k+1) - \{y_1(t_3,t_1) - y_1(t,t_1)\}$$

$$= -(t-(s-k+1)) - \{(t_3-t_1)-(t-t_1)\}$$

$$= (s-(t_3+k-1)) \geq 0, \quad \text{since } s \in [t_3+k-1, t_4+k-1].$$

This gives us that $\Delta G(t,s)$ is nondecreasing for $t \in [s+k-1, t_4+2]$. If we could show that $\Delta G(t_4+2,s) \leq 0$ then $G(t,s)$ would be a decreasing function on $[s+k-1, t_4+3]$. Then, if $G(t_4+3,s) > 0$ we would have that $G(t,s) > 0$ for $t \in [s+k-1, t_4+3]$. So, we consider

$$\Delta G(t_4+2,s) = -\begin{vmatrix} y_2(t_4+2,s-k+1) & 1 & y_1(t_4+2,t_1) & y_2(t_4+2,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= -y_2(t_4+2,s-k+1) + \begin{vmatrix} 1 & y_1(t_4+2,t_1) & y_2(t_4+2,t_1) \\ 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 1 & y_1(t_3,t_1) \end{vmatrix}.$$

Examining the first term we have $(t_4+2-s+k) > (t_4+1-s+k) \geq (t_4+2-(t_4+k-1)+k) = 3 > 0$, which gives us that $-y_2(t_4+2,s-k+1) < 0$. We now

consider the determinant term. Let the function $h(r)$, be the determinant term with t_1 replaced by r. So

$$h(r) = \begin{vmatrix} 1 & y_1(t_4+2,r) & y_2(t_4+2,r) \\ 1 & y_1(t_2,r) & y_2(t_2,r) \\ 0 & 1 & y_1(t_3,r) \end{vmatrix}, \quad \text{which gives}$$

$$\Delta_r h(r) = -\begin{vmatrix} 1 & 1 & y_2(t_4+2,r) \\ 1 & 1 & y_2(t_2,r) \\ 0 & 0 & y_1(t_3,r) \end{vmatrix} - \begin{vmatrix} 1 & y_1(t_4+2,r) & y_1(t_4+2,r) \\ 1 & y_1(t_2,r) & y_1(t_2,r) \\ 0 & 1 & 1 \end{vmatrix}$$
$$= 0.$$

Thus $h(r)$ is a constant. Evaluating h at t_2 gives us

$$h(t_2) = \begin{vmatrix} 1 & y_1(t_4+2,t_2) & y_2(t_4+2,t_2) \\ 1 & y_1(t_2,t_2) & y_2(t_2,t_2) \\ 0 & 1 & y_1(t_3,t_2) \end{vmatrix} = \begin{vmatrix} 1 & y_1(t_4+2,t_2) & y_2(t_4+2,t_2) \\ 1 & 0 & 0 \\ 0 & 1 & y_1(t_3,t_2) \end{vmatrix}$$
$$= -\{y_1(t_4+2,t_2)y_1(t_3,t_2) - y_2(t_4+2,t_2)\}$$
$$= \frac{(t_4+2-t_2)^{(2)}}{2!} - (t_4+2-t_2)(t_3-t_2)$$
$$= \frac{(t_4+2-t_2)(t_4+1-t_2)}{2!} - (t_4+2-t_2)(t_3-t_2)$$
$$= \frac{1}{2}(t_4+2-t_2)\{(t_4-t_3)+1-(t_3-t_2)\}.$$

Thus we have that $h(t_2) < 0$, since by hypothesis $(t_4-t_3)+1 < (t_3-t_2)$. Hence, $h(r) \le 0$ and so our determinant is ≤ 0. This gives us that $\Delta G(t_4+2,s) < 0$, so we have that $G(t,s)$ is decreasing in t on $[s+k-1, t_4+3]$ provided that $(t_4-t_3)+1 < (t_3-t_2)$. Hence if $G(t_4+3,s) > 0$, then $G(t,s) > 0$ for all t in $[s+k-1, t_4+3]$.

We now will evaluate $G(t_4+3,s)$. If we consider $G(t_4+3,s)$ as a function of

s, and then take the difference with respect to s and let $\hat{s} = s - k + 1$, we get

$$\Delta_s G(t_4+3,s) = -\begin{vmatrix} -y_2(t_4+3,\hat{s}) & y_1(t_4+3,t_1) & y_2(t_4+3,t_1) & y_3(t_4+3,t_1) \\ 0 & 1 & y_1(t_2,t_1) & y_2(t_2,t_1) \\ 0 & 0 & 1 & y_1(t_3,t_1) \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= y_2(t_4+3, s-k+1) = \frac{1}{2!}(t_4+3-(s-k+1))^{(2)}$$

$$= \frac{1}{2!}(t_4+2-s+k)^{(2)} > 0,$$

since $(t_4+2-s+k) > (t_4+1-s+k) \geq (t_4+1-(t_4+k-1)+k) = 2 > 0$. Thus $G(t_4+3,s)$ is increasing in s for s in $[t_3+k-1, t_4+k-1]$. This gives us that $G(t_4+3, t_3+k-1) \leq G(t_4+3,s)$ for all $s \in [t_3+k-1, t_4+k-1]$. But from Case 2) we know that $G(t_4+3,t_3) > 0$ provided that $(t_2-t_1) > (t_4-t_2)+1$. Hence we have that if $(t_2-t_1) > (t_4-t_2)+1$ and $(t_3-t_2) > (t_4-t_3)+1$ then $G(t,s) > 0$ for all $t \in (t_1, t_4+3]$, $s \in [t_3+k-1, t_4+k-1]$.

Thus, combining all of our cases, we have shown that if we have $(t_2-t_1) > (t_4-t_2)+1$ and $(t_3-t_2) > (t_4-t_3)+1$ then $G(t,s) > 0$ for all $t \in (t_1, t_4+3]$, $s \in [t_1+k, t_4+k-1]$. Hence we have that under the conditions stated, hypothesis (H) holds.

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